Correctness of Work Function Algorithm for Metrical Task Systems

Usual dynamic programming table is constructed, but WFA selects one entry $w_{i+1}(s_{i+1})$ in each row.

This is a state *j* that minimizes $w_{i+1}(j) + d(s_i, j)$ where s_i is the WFA choice in previous row

but must also be a state *j* with the property $w_{i+1}(j) = w_i(j) + r_{i+1}(j)$.

Theorem: Since there exists *j'* that satisfies the first prop., then there exists *j* that satisfies both props.

Proof:

Used later:

$$(\forall x)\left(w_{i+1}(x) \le w_i(x) + r_{i+1}(x)\right)$$

Suppose *j'* satisfies the first property and *j* is its "predecessor" in the DP table:

$$w_i(j) + r_{i+1}(j) + d(j, j') = w_{i+1}(j')$$

Now add $d(j, s_i) - d(j, j')$ to both sides:

$$w_i(j) + r_{i+1}(j) + d(j, s_i) = w_{i+1}(j') + d(j, s_i) - d(j, j')$$

Since $w_{i+1}(j) \le w_i(j) + r_{i+1}(j)$ and $d(j, s_i) - d(j, j') \le d(j', s_i)$:

$$w_{i+1}(j) + d(j, s_i) \le w_i(j) + r_{i+1}(j) + d(j, s_i) = w_{i+1}(j') + d(j, s_i) - d(j, j') \le w_{i+1}(j') + d(j', s_i)$$

But due to the hypothesis, $w_{i+1}(j') + d(j', s_i)$ minimizes, so:

$$w_{i+1}(j') + d(j', s_i) \le w_{i+1}(j) + d(j, s_i) \le \cdots \le w_{i+1}(j') + d(j', s_i)$$

Which gives the first property for *j*:

$$w_{i+1}(j') + d(j', s_i) = w_{i+1}(j) + d(j, s_i)$$

But from the $\cdots$ we obtain the second property:

$$w_{i+1}(j) + d(j, s_i) = w_i(j) + r_{i+1}(j) + d(j, s_i)$$

$$w_{i+1}(j) = w_i(j) + r_{i+1}(j)$$