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Multi-unit auctions with budget limits ☆

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ABSTRACT

We study multi-unit auctions for bidders that have a budget constraint, a situation very common in practice that has received relatively little attention in the auction theory literature. Our main result is an impossibility: there is no deterministic auction that (1) is individually rational and dominant-strategy incentive-compatible, (2) makes no positive transfers, and (3) always produces a Pareto optimal outcome. In contrast, we show that Ausubel's "clinching auction" satisfies all these properties when the budgets are public knowledge. Moreover, we prove that the "clinching auction" is the *unique* auction that satisfies all these properties when there are two players. This uniqueness result is the cornerstone of the impossibility result. Few additional related results are given, including some results on the revenue of the clinching auction and on the case where the items are divisible.

1. Introduction

The starting point of almost all of auction theory is the set of players' *valuations*: how much value (measured in some currency unit) does each of them assigns to each possible outcome of the auction. When attempting actual implementations of auctions, a mismatch between theory and practice emerges immediately: *budgets*. Players often have a maximum upper bound on their possible payment to the auction – their budget.[3] A concrete example is Google's and Yahoo's ad-auctions, where budgets are an important part of a user's bid, and are perhaps even more real for the users than the rather abstract notion of a valuation.[4] Other examples with identical items include standard Treasury, spectrum or electricity auctions. Clearly budgets play an important role in all these as well. Addressing budgets properly breaks down the usual results from the quasi-linear setting, and in particular the VCG mechanism loses its incentive compatibility. The design of dominant-strategy incentive-compatible mechanisms becomes significantly more involved.

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[3] The nature of what this budget limit means for the bidders themselves is somewhat of a mystery since it often does not seem to simply reflect the true liquidity constraints of the bidding firm. There seems to be some risk control element to it, some purely administrative element to it, some bounded-rationality element to it, and more.
[4] See the paper of Nisan et al. (2009) for a more detailed discussion on Google's auction structure.

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Our model in this paper is simple: There are m identical indivisible units for sale, and each bidder i has a private value v_i for each unit, as well as a budget limit b_i on the total amount he may pay. We also consider the limiting case where m is large by looking at auctions of a single infinitely divisible good. Our assumption is that bidders are utility-maximizers, where i's utility from acquiring x_i units (or a fraction of x_i of the good, in the infinitely divisible good case) and paying p_i is $u_i = x_i \cdot v_i - p_i$, as long as the price is within budget, $p_i \leqslant b_i$, and is negative infinity (infeasible) if $p_i > b_i$.[5] Thus the utility is linear in the payment only for outcomes in which the payment is at most the budget. This makes our setting non-quasi-linear.

We study the fundamental question of how to produce efficient allocations in an incentive-compatible way, using the most basic solution concept of dominant-strategies. As the setting is not quasi-linear, allocational efficiency is not uniquely defined since different outcomes are preferred by different players.[6] We thus focus at a weak efficiency requirement: Pareto optimality, i.e., outcomes where it is impossible to strictly improve the utility of some players without hurting those of others (this does not imply agreement among players over allocations – different players may prefer different Pareto optimal allocations). There exist *many* Pareto optimal allocation rules, and we wish to identify those that are implementable in dominant-strategies. In the sequel we always use the term "incentive compatibility" to denote dominant strategy incentive compatibility (this is usually termed "truthfulness" in the computer science literature).

1.1. Main results

Our main result is an impossibility: *there is no deterministic, incentive compatible, and Pareto optimal auction*, for any finite number $m > 1$ of units of an indivisible good and any $n \geqslant 2$ number of players.[7] The cornerstone of the analysis is a characterization result for the case where budgets are public information. For this case we show that Ausubel's "clinching auction" (Ausubel, 2004) is Pareto optimal and incentive compatible.[8] Moreover we show that the clinching auction is the *unique* (up to tie-breaking) auction that satisfies the above properties, when there are exactly two bidders. The assumption of public budgets was made many times before us, e.g. by Laffont and Robert (1996) and in Maskin (2000), and thus we do not wish to argue that private budgets are more plausible than public budgets. On the contrary, we view the second result as a useful *positive* result, which completely pin-points the (only) possible incentive compatible mechanism that is also Pareto optimal.

We emphasize that the main point of the uniqueness result is *not* that payments are unique for the allocation rule of the clinching auction. Indeed this would easily follow from the Revenue Equivalence Theorem (which in turn follows from the Envelope Theorem, as shown in Milgrom and Segal, 2002). Rather, the main point is that the allocation rule itself is the *unique allocation rule* that satisfies the above properties, most notably incentive compatibility and Pareto optimality. In contrast to the quasi-linear setting, where welfare-maximization is the only Pareto optimal rule (regardless of the other properties), in our setting there exist many deterministic allocation rules that are Pareto optimal, individually rational, and with no-positive-transfers. There also exist many deterministic dominant-strategy mechanisms that are individually rational and with no-positive-transfers (even with private budgets). We show that it is the combination of incentive compatibility and Pareto optimality that yields the impossibility for private budgets, and the uniqueness for public budgets.

To see the type of effects that budget limitations create in this setting, recall that Ausubel's auction gradually increases a price parameter, and bidders keep decreasing their demands for items at this price. Whenever the combined demand of the other bidders decreases strictly below available supply, bidder i "clinches" the remaining quantity at the current price. Thus different amounts of units are acquired by bidders at different prices, and the total payment of a bidder is the sum of the prices of all units that he clinched throughout the auction. Ausubel shows that, in the quasi-linear setting, this auction yields exactly the VCG outcome and is thus incentive compatible. The key property for incentive compatibility is that the demands for future items are fixed and independent of the prices at which previous items were acquired. With budgets, this property no longer holds, and demand for future items changes as a function of the remaining budget. If bidder A slightly delays to report a demand decrease, bidder B will pay as a result a slightly higher price for his acquired items, which reduces his future demand. In turn, the fact that bidder B now has a lower demand implies that bidder A pays a lower price for future items, and the contradiction to incentive compatibility becomes evident. Thus with private budgets this auction is no longer incentive compatible, and our analysis implies that this difficulty is inherent to all Pareto optimal allocation schemes. This seems to be the most common strategic problem that budgets introduce, see e.g. Benoit and Krishna (2001) and Brusco and Lopomo (2008). With public budgets (and private values), on the other hand, this manipulation is not possible. Moreover, for two bidders, the clinching auction is the *unique* incentive compatible and Pareto optimal auction.

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[5] This model naturally generalizes to any type of multi-item auction: bidders have a valuation $v_i(\cdot)$ and a budget b_i, and their utility from acquiring a set S of items and paying p_i for them is $v_i(S) - p_i$ as long as $p_i \leqslant b_i$ and negative infinity if the budget has been exceeded $p_i > b_i$. It is interesting to note that the "demand-oracle model" (see e.g. Blumrosen and Nisan, 2007) represents such bidders as well. Analyzing combinatorial auctions with budget limits, even in simple settings such as additive valuations, is clearly a direction for future research.

[6] In quasi-linear settings any Pareto optimal outcome must optimize the "social welfare" – the sum of bidders valuations – and thus efficiency is justifiably interpreted as maximizing social welfare.

[7] This theorem assumes "individual rationality" and "no positive transfers", i.e. that bidders are not paid by the auction nor do they pay more than their value or budget. Without this, the budget limits can be easily side-stepped, e.g., by using a VCG mechanism that pays losers the total value of the others.

[8] The original paper (Ausubel, 2004) makes several initial observations regarding the potential usefulness of the clinching auction when players have budgets, e.g. in the last paragraph of p. 1457 and in footnote 8.

To complete the picture we also analyze some revenue properties of the clinching auction in our setting with budgets. We show that, as the number of items increases and the "dominance" of each bidder decreases, the revenue of this mechanism approaches the revenue of a non-discriminatory monopoly, that knows the values and budgets of the players and determines a single unit-price in order to maximize revenue.

While in the quasi-linear setting, exact formulas for the outcome of the auction can be described (this is essentially the VCG mechanism), in our setting it is quite hard to come up with a parallel closed-form solution, especially in the infinitely divisible good case for which the auction is a continuous time process. (This once again demonstrates the relative flexibility of ascending auctions versus direct mechanisms when one slightly changes the model.) Nevertheless we present exact closed-form descriptions for an infinitely divisible item and two players. These were certainly surprising for us, as they do not seem to resemble any previously considered auction format. In all cases, once the exact form is found, it is a straight forward exercise to verify incentive compatibility and Pareto optimality. For example, if both players have equal budgets, i.e. w.l.o.g. $b_1 = b_2 = 1$ and $v_1 \leqslant v_2$, then if $\min(v_1, v_2) \leqslant 1$ then the high-value player gets everything and pays the second highest value, and otherwise, the low-value player gets $1/2 - 1/(2 \cdot v_1^2)$ and pays $1 - 1/v_1$ and the high-value player gets $1/2 + 1/(2 \cdot v_1^2)$ and pays 1. This unfamiliar format has an underlying reasoning that we explain in the body of the paper. In parallel to the indivisible case, we show for the divisible case as well that when budgets are public, this auction is the unique *anonymous* Pareto optimal and incentive compatible deterministic auction. In a follow-up to our work, Bhattacharya et al. (2010) further analyze the divisible case, showing additional interesting properties. For example, if budgets are private, then the only profitable manipulation is to over-state one's budgets.

The impossibility for private budgets crucially depends on the assumption that players demand *multiple* items. Indeed, several recent works describe positive results for unit-demand players with budgets. For example, Aggarwal et al. (2009) show that an extension of the Demange–Gale–Sotomayor ascending auction is incentive compatible and Pareto optimal. Hatfield and Milgrom (2005) study a more abstract unit-demand model for players with non-quasi-linear utilities that generalizes both the Gale–Shapley stable-matching algorithm as well as the Demange–Gale–Sotomayor ascending auction, showing incentive compatibility and (in the context of our setting) Pareto optimality. Ashlagi et al. (2010) extend the generalized English auction to settings with budget-constraints, again showing incentive compatibility and Pareto optimality.

Another important assumption that is technically being used in our impossibility proof is the deterministic nature of the mechanisms. Bhattacharya et al. (2010) rely on the fact that understating the true budget is not profitable in the divisible case to design an incentive compatible and Pareto optimal *randomized* mechanism for the divisible case. It remains an interesting open question whether there exists a randomized mechanism that satisfies a randomized version of our requirements for our main setup with indivisible items.

If budgets are all zero (i.e., no payments are possible), any possible assignment of the items to the players in which all items are assigned is Pareto optimal, since players' preferences take a very simple form (namely more items are always strictly better). Thus there exist many incentive compatible and Pareto optimal mechanisms – all those that arbitrarily choose an outcome, independent of the players' values. In this context we note that this setting of no payments is usually termed "multi-unit assignment", and when players' preferences over subsets of items belong to a richer domain, the unique incentive compatible and non-bossy mechanism is the Sequential Dictatorship mechanism (see for example Hatfield, 2009 and Budish and Cantillon, 2010). In light of these works, an additional contribution of our work is to strengthen and make more precise this known conceptual clash between incentive compatibility and Pareto optimality. In particular, while possibility results for the case of zero budgets (mainly dictatorship) and for the case of infinite budgets (the VCG mechanism) are long known, our results show that in our context there is nothing special about these two extremes, it is the fact that budgets are public that enables the existence of incentive-compatible and Pareto optimal mechanisms.

1.2. Related literature

Previous literature on auctions with budgets focus on several different directions. A first branch of works (Che and Gale, 1998; Benoit and Krishna, 2001) analyzes how budgets change the classic results on "standard" auction formats, showing for example that first-price auctions raise more revenue than second-price auctions when bidders are budget-constrained, and that the revenue of a sequential auction is higher than the revenue of a simultaneous ascending auction. A second branch of works (Laffont and Robert, 1996; Pai and Vohra, 2008) constructs single-item auctions that maximize the seller's revenue, and a third branch (Maskin, 2000) considers the problem of "constrained efficiency": maximizing the expected social welfare under Bayesian incentive compatibility constraints. A fourth branch (Borgs et al., 2005; Abrams, 2006), taken by the computer science community, tries to design dominant-strategy incentive-compatible multi-unit auctions that approximate the optimal revenue.

Ausubel's clinching auction inspired many follow-up studies in various different directions. For example, Perry and Reny (2005) give a closely related auction for the case of interdependent values. Kagel and Levin (2001) study Ausubel's auction in field experiments. Bae et al. (2008) extend the clinching auction to allow for one bidder with a valuation that is not marginally decreasing. Mishra and Parkes (2009) design a "dual" descending price (Dutch) auction that reaches the VCG outcome and relies on ideas similar to Ausubel's clinching technique. Ausubel (2006) extends his own auction to a setting with heterogeneous items.

The rest of the paper is organized as follows. We start with basic definitions and preliminary propositions in Section 2. The clinching auction (adjusted for our setting) is defined in Section 3, where we also analyze its basic properties: Pareto optimality and incentive compatibility. Section 4 shows the uniqueness of this auction. Relying on this, Section 5 then proves the impossibility result for private budgets. Section 6 discusses some properties of the revenue of the clinching auction for players with budgets, and Section 7 describes the closed-form mechanism for a divisible item.

2. Preliminaries and notation

2.1. Outcomes

We will be considering auctions of m identical indivisible items as well as the limiting case of a single infinitely divisible good.

We have n bidders, where each bidder i has a value v_i for each unit he gets, and has a budget limit b_i on his payment. Rather than explicitly declaring a bidder's utility of going over-budget to be negative infinity, we will equivalently directly declare such cases to be infeasible.

Definition 2.1. An outcome (x, p) is a vector of allocated quantities $x_1, \ldots, x_n$ and a vector of payments $p_1, \ldots, p_n$ with the following properties:

1. (*Feasibility*) In the case of finite m, x_i must be a non-negative integer and $\sum_i x_i \leqslant m$. In the case of an infinitely divisible good, x_i must be non-negative real and $\sum_i x_i \leqslant 1$.
2. (*No Positive Transfers* (*NPT*)) $\sum_i p_i \geqslant 0$.
3. (*Individual Rationality* (*IR*)) $p_i \leqslant x_i \cdot v_i$.
4. (*Budget limit*) $p_i \leqslant b_i$.

Our "no positive transfers" property is weak, in the sense that it allows the outcome to hand in payments to players. The only restriction is that, overall, the auctioneer does not hand money to the players. The weaker definition strengthens the impossibility and uniqueness results, and in addition we note that all the auctions we describe actually satisfy the stronger version of the "no positive transfers" property, where for every player i we have $p_i \geqslant 0$, i.e., no player gets money from the auction.[9] The fact that we require the quantities $x_1, \ldots, x_n$ to be integers also implies that we require deterministic outcomes.

2.2. Auctions and incentives

We will be formally considering only direct revelation auctions where bidders submit their value and budget to the auction, that based on the types $v_1, \ldots, v_n$ and $b_1, \ldots, b_n$ calculates the outcome $x_1, \ldots, x_n$ and $p_1, \ldots, p_n$. Our auctions have a very natural interpretation as dynamic ascending auctions,[10] but for simplicity we will just consider the auction mechanism as a black-box direct-revelation one.

Definition 2.2. A mechanism is incentive compatible if for every $v = (v_1, \ldots, v_n)$, $b = (b_1, \ldots, b_n)$, and every possible manipulation v_i' and b_i', we have that $u_i = x_i \cdot v_i - p_i \geqslant x_i' \cdot v_i - p_i' = u_i'$, where (x_i, p_i) are the allocation and payment of i when he declares (v_i, b_i) and (x_i', p_i') are the allocation and payment of i when he declares (v_i', b_i') (while the other declarations are fixed at (v_{-i}, b_{-i})).

A mechanism is incentive compatible for the case of publicly known budgets if the definition above holds for all v_i', having fixed $b_i' = b_i$.

2.3. Pareto optimality

We start with the classic notion of Pareto optimality:

Definition 2.3. An outcome $\{(x_i, p_i)\}$ is Pareto optimal if for no other outcome $\{(x_i', p_i')\}$ are all players better off, $x_i' v_i - p_i' \geqslant x_i v_i - p_i$, including the auctioneer $\sum_i p_i' \geqslant \sum_i p_i$, with at least one of the inequalities strict.

[9] The weak version is necessary for the uniqueness result. Consider, for example, the following mechanism for one item and two players with infinite budgets: the item is allocated to player 1 if $v_1 > 0$, and otherwise to player 2. No payments are made. One can verify that this is incentive compatible. It is also Pareto optimal if one requires the strong NPT property, since if $v_2 > v_1 > 0$, the only outcome that Pareto-dominates the one chosen by the mechanism is an outcome in which player 1 receives a payment of v_1, and player 2 receives the item and pays v_1. The sum of payments here is 0, so with weak NPT the outcome is not Pareto optimal, and the mechanism can be ruled out.

[10] As usual, the solution concept for the iterative version is ex-post-Nash.

Recall that an outcome requires by definition that payments will not exceed budgets, hence a player's utility in some outcome (x, p) is $x_i v_i - p_i$. The definition of an outcome also requires this utility to be non-negative.

In our setting, the notion of Pareto optimality is equivalent to the following condition that is much easier to work with, and that essentially states that no player can re-sell the items he received and make a profit:

Proposition 2.4. *An outcome* $\{(x_i, p_i)\}$ *is Pareto optimal in the infinitely divisible case if and only if* (a) $\sum_i x_i = 1$, *i.e. the good is completely sold, and* (b) *for all i such that* $x_i > 0$ *we have that for all j with* $v_j > v_i$, $p_j = b_j$, *i.e. a player may get a non-zero outcome only if all higher value players have exhausted their budget.*

For example, the outcome that awards all items to a buyer with highest value, and requires no payment, is Pareto optimal, and indeed the two requirements of the claim hold (the second requirement holds in an empty way). The proof is given in Appendix A. A similar property is equivalent to Pareto optimality also in the case of finite m (the proof is similar to the proof of the previous claim):

Proposition 2.5. *An outcome* $\{(x_i, p_i)\}$ *is Pareto optimal in the case of finite m if and only if* (a) $\sum_i x_i = m$, *i.e., all the units are sold, and* (b) *for all i such that* $x_i > 0$ *we have that for all j with* $v_j > v_i$, $p_j > b_j - v_i$, *i.e. a player may get a non-zero outcome only if there is no player with higher value that has larger remaining budget.*

2.4. Warmup: The proportional share auction

Recall that our main goal is to show the impossibility of constructing a mechanism that is Pareto optimal and incentive compatible when budgets are private. Before that, we wish to point out that if values are guaranteed to be sufficiently large relative the budgets, a simple mechanism exists:

Definition 2.6. The proportional share auction for an infinitely divisible good allocates to each bidder i a fraction $x_i = b_i / \sum_j b_j$ of the good and charges him his total budget $p_i = b_i$.

We prove in Appendix B:

Proposition 2.7. *Let* $\alpha_i = b_i / \sum_j b_j$ *be the budget share of player i. The proportional-share auction with* $x_i = b_i / \sum_j b_j$ *and* $p_i = b_i$ *is Pareto optimal and incentive compatible in the range* $v_i \geqslant \sum_j b_j / (1 - \alpha_i)$ *for all i.*

3. The clinching auction for players with budgets

We formally describe the clinching auction for players with *public* budgets, and show that it satisfies Pareto optimality, individual rationality, and incentive compatibility. The formal auction we describe is a direct mechanism whose outcome is chosen to be the outcome of Ausubel's clinching auction, when budget-constrained players bid sincerely in it. Ausubel's auction gradually increases a price parameter, and bidders keep decreasing their demands for items at this price. Whenever the combined demand of the other bidders decreases strictly below available supply, bidder i "clinches" the remaining quantity at the current price. Thus different amounts of units are acquired by bidders at different prices, and the total payment of a bidder is the sum of the prices of all units that he clinched throughout the auction.

Before we begin the formal discussion, it might be useful to point out a subtle but important difference between the course of the clinching auction in the quasi-linear setting versus the budget setting: In the quasi-linear setting the demand curves of the bidders remain static, unchanged, throughout the course of the auction (the supply of-course changes). In the budget setting, *demands themselves change*, as previous clinching affect remaining budget, that in turn affects future demand. So demand as well as supply change. To emphasize this effect that the change in setup has on the clinching auction, we add the words "with budgets" to its name.

Formally, the auction keeps for every player i the current number of items q_i already allocated to i, the current total price for these items p_i, and his remaining total budget $B_i = b_i - p_i$. The auction also keeps the global unit-price p and the global remaining number of items q. The price p gradually ascends as long as the total demand is strictly larger than the total supply, where the demand of player i is defined by:

$$D_i(p) = \begin{cases} \lfloor \frac{B_i}{p} \rfloor & v_i > p, \\ 0 & \text{otherwise.} \end{cases}$$

If we were to keep the price ascending until total demand would be smaller or equal to the number items, and only then allocate all items according to the demands, then a player could sometimes gain by performing a "demand reduction", thus harming incentive compatibility. Instead, following Ausubel's method, we allocate items to player i as soon as the total demand of the *other* players decreases strictly below the number of currently available items, q. In particular, if at some price p we have $x = q - \sum_{j \neq i} D_j(p) > 0$ then we allocate x items to player i for a unit price p. At this point in the auction, the relevant variables are updated as follows: $q_i \leftarrow q_i + x$, $p_i \leftarrow p_i + p \cdot x$, $B_i \leftarrow B_i - p \cdot x$, and $q \leftarrow q - x$. This will ensure incentive compatibility. The global picture of such an auction is:

The clinching auction with budgets (preliminary version):

1. Initialize all running variables: $p \leftarrow 0$, $q \leftarrow m$, $q_i \leftarrow 0$, $p_i \leftarrow 0$, $B_i \leftarrow b_i$.
2. While $\sum_i D_i(p) > q$,
   (a) If there exists a player i such that $D_{-i}(p) = \sum_{j \neq i} D_j(p) < q$ then allocate $q - D_{-i}(p)$ items to player i for a unit price p. Update all running variables, and repeat.
   (b) Otherwise increase the price p, recompute the demands, and repeat.
3. Otherwise (hopefully $\sum_i D_i(p) = q$): allocate to each player his demand, at a unit-price p, and terminate.

Note that step 2(a) does not change the amount of excess demand, since both the total demand and the total supply are reduced by the same quantity (the number of items that player i gets). Therefore the only factor that affects the excess demand is the price; as the price ascends the total excess demand decreases. Thus, one would hope that when we reach step 3 we would indeed get $\sum_i D_i(p) = q$, which will enable us to allocate all items at the end (a necessary condition for achieving Pareto optimality). However clearly this is not quite the case, because the demand functions are not continuous. The demand drops integrally, by definition, and may drop by several items at once. In particular, there are two potentially problematic change points: when the price reaches the value v_i, and when the price reaches the remaining budget B_i. The latter point is identified by using:

$$D_i^+(p) = \lim_{x \to p^+} D_i(x),$$

as, for $p = B_i < v_i$, we have $D_i(p) > 0$ and $D_i^+(p) = 0$. Similarly, the former point is identified by using:

$$D_i^-(p) = \lim_{x \to p^-} D_i(x),$$

as, for $p = v_i \leqslant B_i$, we have $D_i^-(p) > 0$ and $D_i(p) = 0$. We modify the above definition of the auction to use these more refined conditions: (1) the excess demand is computed using $D_i^+(p)$, since this ensures that we do not terminate with a price that is just a bit higher than the remaining budget of a player to whom we wish to allocate one last item, and (2) just before termination, if we are left with some non-allocated items, then this must have happened because the final price reached the value of some players (for such a player i we have $D_i^-(p) > 0$ and $D_i(p) = 0$), which caused an abrupt decrease in his demand. These players are indifferent between receiving or not receiving an item, and so we can allocate to them all remaining items.

The clinching auction with budgets (complete version):

1. Initialize all running variables: $p \leftarrow 0$, $q \leftarrow m$, $q_i \leftarrow 0$, $p_i \leftarrow 0$, $B_i \leftarrow b_i$.
2. While $\sum_i D_i^+(p) > q$,
   (a) If there exists a player i such that $D_{-i}^+(p) = \sum_{j \neq i} D_j^+(p) < q$ then allocate $q - D_{-i}^+(p)$ items to player i for a unit price p. Update all running variables (including the allocated and available quantities, the remaining budgets, and the current demands), and repeat.
   (b) Otherwise increase the price p, recompute the demands, and repeat.
3. Otherwise ($\sum_i D_i^-(p) \geqslant q \geqslant \sum_i D_i^+(p)$):
   (a) For every player i with $D_i^+(p) > 0$, allocate $D_i^+(p)$ units to player i for a unit-price p and update all running variables.
   (b) While $q > 0$ and there exists a player i with $D_i(p) > 0$, allocate $D_i(p)$ units to player i, for a unit-price p, and update the running variables.
   (c) While $q > 0$ and there exists a player i with $D_i^-(p) > 0$, allocate $D_i^-(p)$ units to player i, for a unit-price p.
   (d) Terminate.

We note that there may be multiple players at step 2(a) that can clinch item(s). In this case we arbitrarily pick one of them, and continue. The other players will subsequently clinch items in the next iterations, before the price increases. In particular, if there exist two players i, j such that at price p, $D_{-i}^+(p) < q$ and $D_{-j}^+(p) < q$, and i is chosen to clinch first (say he receives x items), then at the next iteration the number of items has decreased by x, and $D_{-j}^+(p)$ has also decreased by x since it includes i's demand that was reduced by x. Therefore the inequality $D_{-j}^+(p) < q$ still holds, and now j will clinch. Thus there cannot be any "buying frenzies" during the auction – any player that can clinch a certain number of items at a certain step will eventually receive all these items, before the price will increase.

Let us consider a short example to illustrate the course of the auction. Suppose three items and three players with $v_1 = \infty$, $b_1 = 1$, $v_2 = \infty$, $b_2 = 1.9$, $v_3 = 1$, $b_3 = 1$. When the price is below 0.5, each player demands at least two items, and so, for every player, the other players demand more than three items. Therefore no allocations will take place, and the price will keep ascending. At $p = 0.5$, $D_1^+(0.5) = D_3^+(0.5) = 1$ (note that $D_1(0.5)$ and $D_3(0.5)$ are still 2). Thus, player 2 "clinches" one item for a price 0.5. Immediately after that, the demand of player 2 is updated to be 2. The available number of items is 2, and so no player can get any items. At a price 0.7 the demand of player 2 reduces to 1, but this still does not enable the auction to allocate any item to any player. The price keeps ascending until $p = 1$. At this point, $D_1^+(1) = 0, D_2^+(1) = 1, D_3^+(1) = 0$, and so the total demand reduces to be strictly below the number of available items (which is still 2). Thus we enter step 3. In 3(a) player 2 gets one item and in 3(b) player 1 gets one item. Note that we do not allocate any item to player 3, though $D_3^-(1) = 1$. Indeed, moving an item from 2 to 3, for example, will violate Pareto optimality.

The individual rationality and incentive compatibility of the clinching auction is almost immediate: individual rationality follows since a player's payment for every clinched item is not larger than his reported value, and his overall payment is at most his budget. Incentive compatibility is also easy to show, since the value declaration is equivalent to a simple decision of when to quit the auction. Since any item clinched in a price lower than the true value strictly increases the player's utility, and any item clinched in a price larger than the true value strictly decreases the player's utility, it follows that the player's utility is maximized by quitting exactly when the price reaches his value, which implies incentive compatibility. Proving Pareto optimality is a slightly more involved task. We first show that all items are always allocated:

Claim 3.1. *The clinching auction always allocates all items.*

Proof. Define $D(p) = \sum_i D_i(p)$ and define $D^+(p)$ and $D^-(p)$ similarly. Observe that these three functions are monotone non-increasing, and that $D^-(p) = D(p) = D^+(p)$ for any continuity point of $D(p)$. Moreover, if p^* is a discontinuity point of $D(p)$ and $D^+(p) > q$ for any $p < p^*$ then $D^-(p^*) \geqslant q$.

Suppose that the auction enters step 3 at a price p^*. We wish to argue that $D^-(p^*) \geqslant q$. Indeed, for any $p < p^*$, at the beginning of step 2 we had $D^+(p) > q$, and after step 2(a) this inequality is maintained (since if we allocate Δ units to player i then the total demand and the number of available items both drop by Δ). Therefore after step 2(b) we have $D^+(p) \geqslant q$ (if p is a continuity point) or $D^+(p) < q$ and $D^-(p) \geqslant q$ (if p is a discontinuity point). In any case, if the auction enters step 3 then $D^-(p^*) \geqslant q$, and the claim follows. □

Claim 3.2. *The clinching auction satisfies Pareto optimality.*

Proof. We will check the condition of Proposition 2.5. We already showed property (a) ($\sum_i x_i = m$) and it remains to show property (b). Fix any two players i and j. We need to verify that, if j received at least one item, then i's remaining budget at the end of the auction is smaller than j's value. Consider the last price p at which player j received an item.

First suppose that p is not the price that ended the auction. In this case (step 2(a)), since j received an item, the auction rules imply that $D_{-j}^+(p)$ exactly equals the number of items left after player j was allocated his items. Since the auction allocates all items, and since it is IR, we get that each player $i \neq j$ received after price p exactly $D_i(p)$, his demand at p. In particular, this means that the remaining available budget of i is at most p (otherwise the demand of i at p was higher – he could have bought one more item at a price lower than his value). On the other hand, $v_j > p$, since j demanded items at p, and we are done.

Now suppose that p is the price at which the auction ended. The auction rules imply that if i had $D_i^+(p) > 0$ then he received all this demand, and so by the same argument as above he does not have any remaining budget to buy an item from j. A second case is $D_i^+(p) = 0$ and $D_i(p) > 0$. This implies that the remaining budget of player i at this step is $B_i = p$. If player i received his demand $D_i(p)$ then the argument of above still holds. If not, it must be that player j received her items in step 3(a) or 3(b) (but not in 3(c), since not all players in 3(b) were awarded their demand). Thus $D_j(p) > 0$ hence $v_j > p = B_i$ and a Pareto improvement cannot take place. The last case is $D_i(p) = 0$ and $D_i^-(p) > 0$. Hence $p = v_i$, and since $v_j \geqslant p$ this again rules out the possibility of a Pareto improvement. □

An interesting future work would be to study other more complex settings with complementarities or with non-identical items. A first step in this direction has been performed very recently by Fiat et al. (2011), studying single valued combinatorial auctions with budgets, showing how to extend the clinching auction to this setup.

4. Uniqueness of the clinching auction

In this section we show that the ascending clinching mechanism is essentially the only mechanism that is incentive compatible, individually rational, and Pareto optimal for the setting of publicly known budgets. In the next section we utilize this result to show that there is no mechanism if the budgets are private.

Strictly speaking, we do not prove uniqueness for all possible budgets b_1 and b_2, but for "almost" all budgets. This is in a sense the best we can hope for, as, for example, for one item and $b_1 = b_2$ there are indeed multiple possible auctions (which are identical up to tie breaking). The following technical definition attempts to deal with this issue.

Let $S = (S_1, S_2)$ be a partition of $\{1, \ldots, m\}$. Given $b_1, b_2 \geqslant 0$, define $b_i^{k,S}$ recursively, for each $1 \leqslant k \leqslant m$: for $k = m$, $b_1^{m,S} = b_1$, $b_2^{m,S} = b_2$. For each $1 \leqslant k \leqslant m - 1$, if $k \in S_1$ then: $b_1^{k,S} = b_1^{k+1,S}$, $b_2^{k,S} = b_2^{k+1,S} - \frac{b_1^{k+1,S}}{k+1}$. If $k \in S_2$ then: $b_1^{k,S} =$

$b_1^{k+1,S} - \frac{b_2^{k+1,S}}{k+1}$, $b_2^{k+1,S} = b_2^{k+1,S}$. We say that b_1 and b_2 are *S-generic* if for each $1 \leqslant k \leqslant m$ we have that $b_1^{k,S} \neq b_2^{k,S}$. We say that b_1 and b_2 are *generic* if they are S-generic for all S.

Notice that given any b_1 and b_2, a small perturbation will make them generic.

Theorem 4.1. *Let A be a deterministic incentive compatible mechanism for m items and 2 players with known budgets b_1 and b_2 that are generic. Assume that A satisfies Pareto optimality, individual rationality, and no positive transfers. Then if $v_1 \neq v_2$ the outcome of A coincides with that of the clinching auction.*

The proof shows that all mechanisms that satisfy the requirements of the claim have the same outcome. Since the clinching auction satisfies all requirements of the claim, all other mechanisms coincide with it. We start with a useful lemma:

Lemma 4.2. *If $v_j < v_i$ and $v_j \leqslant \frac{b_i}{m}$ then player i receives all items and pays $p_i = m \cdot v_j$ in any deterministic incentive compatible mechanism that satisfies Pareto optimality, individual rationality, and no positive transfers. In this case j's payment, p_j, is exactly zero.*

Proof. First consider the case $v_j < v_i < \frac{b_i}{m}$. In this case if player i receives $x < m$ items then since by IR he pays at most $x \cdot v_i < \frac{m-1}{m} b_i$ he has left enough money to buy an item from player j and pay him $v_j + \epsilon < v_i$, which contradicts Pareto optimality. Thus player i receives all items. Standard monotonicity arguments (see e.g. footnote 12 below) now imply that i receives all items for any $v_i \geqslant \frac{b_i}{m}$ (when $v_j < \frac{b_i}{m}$).

If $v_j = \frac{b_i}{m}$ then for $v_i < \frac{m-1}{m-2} \cdot \frac{b_i}{m}$ it must be that player i receives $x \geqslant m-1$ items, otherwise if $x \leqslant m-2$ then by individual rationality $p_i \leqslant x \cdot v_i \leqslant \frac{(m-1)b_i}{m}$ and $b_i - p_i \geqslant \frac{b_i}{m} = v_j$, and by Lemma 2.5 this contradicts Pareto optimality since $v_i > v_j$. If $x = m-1$ then by monotonicity player i receives $m-1$ items for any value in the interval $(\frac{b_i}{m}, v_i]$, therefore by incentive compatibility his payment p_i is at most $\frac{(m-1)b_i}{m}$. But then again this contradicts Pareto optimality as above. Thus player i receives all items in this case as well.

To prove that the payments are as claimed first suppose that $v_j = 0$. By IR $p_j \leqslant 0$. For any declaration $v_i' > 0$ player i receives all items (as argued above) and pays at most $p_i' \leqslant m \cdot v_i'$. Thus by incentive compatibility if $v_j = 0$ then $p_i \leqslant 0$. No-positive-payments requires $p_i + p_j \geqslant 0$ which implies $p_i = p_j = 0$ for the case $v_i > v_j = 0$.

For a general value v_j, since j receives no items here as well, then incentive compatibility implies $p_j = 0$. Using the standard argument of the second-price auction we finally have that $p_i = m \cdot v_j$, and the claim follows. □

We continue with the main proof. Without loss of generality we assume throughout that $b_1 < b_2$. The proof is by induction on the number of items m, and we start with the base case $m = 1$.

Lemma 4.3. *All mechanisms for one item that satisfy the conditions of Theorem* 4.1 *have the same outcome if $v_1 \neq v_2$.*

Proof. We show that the only possible mechanism is the following: the winner is the player i that maximizes $\min(b_i, v_i)$. The winner pays the mechanism $\min(b_j, v_j)$, where j is the other player, and the loser's payment is exactly zero.[11]

It is easy to verify that the above mechanism satisfies the required properties. We now prove that this is the only possible mechanism. If $\min(v_1, v_2) \leqslant b_1$ then the claim follows from Lemma 4.2. Otherwise assume $v_1, v_2 > b_1$.

We show that player 2 must win the item. First observe that if $v_1 < \min(v_2, b_2)$ then the only Pareto optimal outcome allocates the item to 2 (in the other allocation player 2 can buy the item from 1, and they are both better off). Suppose that there exists some value $v_1' > b_1$ such that 1 wins the item even though $v_2 > b_1$. By feasibility 1's payment in this case is at most b_1, and 1 has positive utility from declaring v_1'. Thus when 1's true value is $b_1 < v_1 < \min(v_2, b_2)$ he can declare v_1' and improve his utility, contradicting incentive compatibility.

Therefore for any $v_2 > b_1$ player 2 must be the winner. Player 1's payment must be exactly zero by incentive compatibility since his payment must be equal to the case when he declares $v_1' < b_1$. This also implies that player 2's payment is the minimal possible value he needs to declare in order to win, i.e. $\min(b_1, v_1)$, and the claim follows. □

We now continue the induction, assuming uniqueness for $m-1$ items, and proving uniqueness for m items. The logic is as follows. We start with some arbitrary mechanism A for m items that satisfies the conditions of Theorem 4.1. We need to show that A is in fact equivalent to the clinching auction. We show this in two parts. First, if $\min(v_1, v_2) \leqslant \frac{b_1}{m}$ (and suppose that $v_j = \min(v_1, v_2)$ and i is the other player), the clinching auction allocates all items to player i, and his payment is mv_j. Player j pays zero in this case. Now, Claim 4.2 shows that A must do the same, since $v_j \leqslant \frac{b_1}{m} \leqslant \frac{b_i}{m}$.

[11] Notice that if b_1 and b_2 are not generic, i.e., $b_1 = b_2$, then indeed this auction is not uniquely defined as if $v_1, v_2 > b_1 = b_2$ we can break ties in favor of either player, resulting in multiple possible outcomes. Also notice that this mechanism is indeed identical to the clinching auction.

The second case, where $v_1, v_2 \geqslant \frac{b_1}{m}$, is more involved. To characterize A's behavior in this domain, we use A to construct a new mechanism A_{m-1} for $m-1$ items *and different budgets*. We will show a one-to-one mapping between the outcome of A for values $v_1, v_2 \geqslant \frac{b_1}{m}$ and the outcome of A_{m-1} for the same values. We will then show that A_{m-1} satisfies the conditions of Theorem 4.1, and therefore the induction assumption implies that A_{m-1} is unique. This in turn implies that A is unique, and therefore it must be the clinching auction (in other words, if there exist A and A' for m items that satisfy the conditions of Theorem 4.1 then there exist A_{m-1} and A'_{m-1} for $m-1$ items that satisfy the conditions of Theorem 4.1 and this contradicts the inductive assumption).

The mechanism A_{m-1} is defined as follows. A_{m-1} works on budgets $b'_1 = b_1$ and $b'_2 = b_2 - \frac{b_1}{m}$. Notice that b'_1 and b'_2 are generic, and that now it is not necessarily true that $b'_1 \leqslant b'_2$. We start by defining A_{m-1} on instances where $v_1, v_2 > \frac{b_1}{m}$: denote the outcome of A for v_1 and v_2 by $(\vec{x}, \vec{p})$, where x_i is the amount that i gets, and p_i is his payment. Let the outcome of A_{m-1} be (x_1, p_1) for player 1 (i.e., as in A), and for player 2 let the outcome be $(x_2 - 1, p_2 - \frac{b_1}{m})$. In particular, observe that given the outcome of A_{m-1} on valuations in this domain, we can deduce the outcome of A on the same valuations. Claims 4.4 and 4.5 below prove that $x_2 \geqslant 1$ and $p_2 \geqslant \frac{b_1}{m}$ and thus the definition is valid. Before showing this we need to complete the definition of A_{m-1} for valuations where $\min(v_1, v_2) \leqslant \frac{b_1}{m}$. In this case we allocate all items to the bidder with the highest value, and his payment is $m-1$ times the value of the other player.

Lemma 4.4. *Let A be a mechanism for m items that is Pareto optimal, individually rational, and incentive compatible. Suppose that $\min(v_1, v_2) > \frac{b_1}{m}$. Then, a player that wins x items pays at least $x \cdot \frac{b_1}{m}$.*

Proof. Suppose by contradiction that there exist (v_1, v_2) in which some player i gets $x \geqslant 1$ items and pays $t < x \cdot \frac{b_1}{m}$. Consider now a different valuation v'_i such that $t/x < v'_i < \frac{b_1}{m}$. By Lemma 4.2 i is allocated no items when he declares are v'_i and the other player declares the same as before. Here i will be better off by declaring v_i instead of v'_i, since he will be allocated x items and will get a positive utility: $x \cdot v'_i - t > 0$, contradicting incentive compatibility. □

Lemma 4.5. *Let A be a mechanism for m items that is Pareto optimal, individually rational, and incentive compatible. Suppose that $v_2 > \frac{b_1}{m}$. Then, player 2 wins at least one item.*

Proof. Suppose that there is a declaration v_1 such that, when the players declare (v_1, v_2), player 1 wins all items. By Lemma 4.4 the payment of player 1 is at least $m \cdot \frac{b_1}{m} = b_1$. His payment is exactly b_1 since this is his budget. By incentive compatibility, in any declaration $v'_1 > \frac{b_1}{m}$ he must still win all items (player 2 still declares v_2). Fix v'_1 such that $\min(v_2, b_2) > v'_1 > \frac{b_1}{m}$. From above we get that player 1 gets all items when the declarations are (v'_1, v_2). However this contradicts Pareto optimality, using Claim 2.5, since $v_2 > v'_1$ but $p_2 = 0 < b_2 - v'_1$. □

Lemma 4.6. *A_{m-1} satisfies the conditions of Theorem 4.1.*

Proof. During the proof we abuse notation a bit and identify the outcome of A with A, and the outcome of A_{m-1} with A_{m-1}. We break the proof into several claims.

Lemma 4.7. *Let A be a mechanism for m items that is Pareto optimal, individually rational, and incentive compatible. Suppose that $v_1 > \frac{b_1}{m}$. Then, if player 2 wins exactly one item he pays exactly $\frac{b_1}{m}$.*

Proof. Fix some v_2 such that, when the declaration is (v_1, v_2), player 2 gets $x_2 = 1$ and pays some p_2. By Claim 4.4, $p_2 \geqslant \frac{b_1}{m}$. Now fix some v'_2 such that $v_2 > v'_2 > \frac{b_1}{m}$. Suppose that in the declaration (v_1, v'_2) player 2 gets x'_2 and pays p'_2. It is well known that incentive compatibility implies that $x'_2 \leqslant x_2$.[12] By Claim 4.5, $x'_2 \geqslant 1$, and therefore we must have $x'_2 = 1$. Incentive compatibility now implies that $p_2 = p'_2$. Therefore we have $v'_2 \geqslant p_2 \geqslant \frac{b_1}{m}$. Since this is true for any $v'_2 > \frac{b_1}{m}$ we get that $p_2 = \frac{b_1}{m}$, as claimed. □

Claim 4.8. *A_{m-1} is individually rational.*

Proof. If $\min(v_1, b_1) \leqslant \frac{b_1}{m}$, then A_{m-1} is a second price auction. Else, if player 1 is allocated no items in A_{m-1}, then he pays nothing, since A is individually rational and 1 gets nothing also in A. Consider the case where player 2 is allocated no items in A_{m-1}. It means that it was allocated exactly one item in A, and by Lemma 4.7 his payment is $\frac{b_1}{m}$ in A, hence in A_{m-1} his payment is 0. □

[12] A short proof, based on the W-MON condition of Bikhchandani et al. (2006), is: from incentive compatibility we have $v_2 \cdot x_2 - p_2 \geqslant v_2 \cdot x'_2 - p'_2$ since when the true type is v_2 the player will not benefit from declaring v'_2. Similarly, $v'_2 \cdot x'_2 - p'_2 \geqslant v'_2 \cdot x_2 - p_2$. Combining, we get $v'_2(x'_2 - x_2) \geqslant p'_2 - p_2 \geqslant v_2(x'_2 - x_2)$, and since $v'_2 < v_2$ it follows that $x'_2 \leqslant x_2$.

Claim 4.9. *A_{m-1} is Pareto optimal.*

Proof. Consider first the case where $v_1, v_2 > \frac{b_1}{m}$. By Claim 2.5, it is enough to show two things: (1) If $v_1 > v_2$ then $p'_1 > b'_1 - v_2$: since A is Pareto optimal then $p_1 > b_1 - v_2$, and since $p'_1 = p_1$ and $b'_1 = b_1$ the claim follows; and (2) If $v_2 > v_1$ then $p'_2 > b'_2 - v_1$, or, equivalently, $v_1 > b'_2 - p'_2$: since A is Pareto optimal then $v_1 > b_2 - p_2$, and since $b'_2 - p'_2 = b_2 - p_2$ the claim follows.

Now consider the case where $\min(v_1, v_2) \leqslant \frac{b_1}{m}$. Let $b'_i = \min(b'_1, b'_2)$. First, observe that we have that if $b'_i = b'_1$ then $\frac{b_1}{m} \leqslant \frac{b'_1}{m-1}$, since $b'_i = b'_1$. For $b'_i = b'_2 = b_2 - \frac{b_1}{m}$, we also have that $\frac{b'_2}{m-1} = \frac{b_2 - \frac{b_1}{m}}{m-1} \geqslant \frac{b_1 - \frac{b_1}{m}}{m-1} \geqslant \frac{b_1}{m}$. Hence in this case, by Lemma 4.2, it is Pareto optimal to allocate all items to the bidder with the highest value, as A_{m-1} indeed does. □

Claim 4.10. *A_{m-1} is incentive compatible.*

Proof. Once again we consider several different cases. Start with the case where $v_1, v_2 > \frac{b_1}{m}$, and suppose player i declares $v'_i > \frac{b_1}{m}$ instead (and is allocated x'_i items and pays p'_i). Clearly, $i \neq 1$, as the allocation and payment of player 1 are the same as in A, and A is incentive compatible. Suppose $i = 2$ is better off declaring v'_2: $v_2(x_2) - p_2 < v_2(x'_2) - p'_2$. Observe that in A we have that: $v_2(x_2 + 1) - (p_2 + \frac{b_1}{m}) < v_2(x'_2 + 1) - (p'_2 + \frac{b_1}{m})$, a contradiction to the incentive compatibility of A.

Suppose that $v_1, v_2 > \frac{b_1}{m}$, and that player i declares $v'_i < \frac{b_1}{m}$ instead. Notice that $x'_i = 0$, so i cannot increase his profit from declaring v'_i.

In the case where $\min(v_1, v_2) \leqslant \frac{b_1}{m}$ player i is not better off declaring $v'_i < \frac{b_1}{m}$, as in this range we are essentially conducting a second price auction, which is incentive compatible.

Finally, suppose $\min(v_1, v_2) \leqslant \frac{b_1}{m}$. Consider player i who declares $v'_i > \frac{b_1}{m}$. Suppose $v_j > \frac{b_1}{m}$, where j is the other player. Observe that if i wins some items, then by Lemma 4.4 j has to pay at least $\frac{b_1}{m}$ for every item he wins, which is more than his value. If $v_j < \frac{b_1}{m}$, then we conduct a second price auction, regardless of what i declares, and this auction is incentive compatible. □

By the induction hypothesis, we have that A_{m-1} is unique. By our discussion, this is enough to prove the uniqueness of A and this concludes the proof of the theorem.

5. An impossibility result for private budgets

Once the public-budgets case is completely analyzed, the impossibility for private budgets follows quite easily.

Theorem 5.1. *There is no deterministic incentive compatible mechanism that satisfies Pareto optimality, individual rationality, and no positive transfers, for private budgets.*

Proof. We first prove the theorem for the case of two players. An auction A for private budgets is also incentive compatible if budgets are public. By our uniqueness result for two players with public budgets, we therefore conclude that the outcome of A must be the same as the outcome of the clinching auction.

Consider two instances of the clinching auction. First, $b_1 = 1$, $v_1 = \infty$, $b_2 = 1 + \sum_{k=2}^{m} \frac{1}{k} - \delta$, $v_2 = \infty$, for some small $\delta > 0$. (δ is chosen to make b_1 and b_2 generic.) For each of the first $m-1$ items, the clinching auction will allocate the item to player 2 and will charge $\frac{1}{k}$ for the k'th item. Then, at the k'th item, player 1's budget is finally larger than player 2's free budget, so player 1 wins the last item with a payment of $1 - \delta$.

Second, $b'_1 = 1 + \epsilon$, for small enough ϵ, and the other parameters are as above. The resulting allocation is the same as above, but player 2 is charged $\frac{1+\epsilon}{k}$ for the k'th item (for $k > 1$). Thus, when the auction allocates the last item, player 2's free budget is smaller than before: $1 - \delta - \sum \frac{\epsilon}{k}$. This is also the payment of player 1.

Therefore player 1 is allocated one item in both cases, but his payment is smaller in the second case, so his utility is larger. Now, as argued in the first paragraph of this proof, A's outcome is the same as the outcome of the clinching auction for both cases. Therefore when the players' types are as in the first case, player 1 can improve his resulting utility from the mechanism A by misreporting his budget to be $b'_1 = 1 + \epsilon$. This will change the outcome of A to be that of the clinching auction for the second case, and will thus increase player 1's utility, which contradicts incentive compatibility.

We now prove the theorem for any number of players. Suppose by contradiction that there exists an auction A_n for $n > 2$ players with private budgets that satisfies all properties of the theorem. Then there is an auction A_2 for two players with private budgets that satisfies all properties of the theorem: upon receiving the declarations of the two players, A_2 adds $n-2$ players that have a budget of zero and a value of zero, and determines the allocation and payments of the two "real" players to be the same as their allocation and payments in A_n with the $n-2$ dummy players. Since A_n satisfies all properties of the claim then A_2 satisfies all properties of the claim as well. This contradicts our previous proof, for the case of exactly two players, and the theorem follows. □

The contradiction in the proof was obtained by reporting a budget which is higher than the true budget. In a follow-up paper, Bhattacharya et al. (2010) show that there are cases where it is also profitable to declare a budget lower than the true budget. They also show that for a *divisible* item, only higher budgets can be profitable deviations.

6. Revenue considerations

Up to now we have discussed the efficiency properties of the clinching auction for players with budgets. We now examine its revenue properties. We will compare the revenue of the clinching auction to the revenue of a non-discriminatory monopoly that knows the budgets and values of the players, and has to determine a single unit-price at which items will be sold. To strengthen our result and simplify the analysis at the same time, we allow the monopoly (but not the mechanism!) to sell also fractions of the good, and not just integer quantities.

The approach of comparing an auction's revenue to the optimal fixed-price revenue was initiated by Goldberg et al. (2006). In the context of auctions with budget limitations it was used by Borgs et al. (2005) and Abrams (2006). In particular, Abrams (2006) showed that the optimal monopoly revenue is always at least half of the optimal *multi-price* revenue, that may charge different prices from different players.[13] Thus, comparing the revenue of the auction to *any other revenue criteria* can yield a ratio which may be smaller by a constant factor of at most 1/2.

To formally define our benchmark for revenue, let a *fractional allocation* be a real vector $x = (x_1, \ldots, x_n)$, where for each i, $x_i \geqslant 0$, and $\sum_i x_i \leqslant m$. Given a fractional allocation x, the *monopoly revenue* from x is $\sum_i x_i \cdot p^*(x)$, where $p^*(x)$ is the largest price that satisfies, for each i with $x_i > 0$, $v_i \geqslant p^*(x)$, and $b_i \geqslant x_i \cdot p^*(x)$. Thus, the monopoly revenue from x is the largest unit-price p that maintains the individual rationality of players for the allocation x with unit-price p. The *optimal monopoly revenue* is the supremum over all fractional assignments x of the monopoly revenue from x.[14] Let x^* be the fractional allocation that obtains this optimal monopoly revenue, and $p^* = p^*(x^*)$. Our analysis uses the following "bidder dominance" parameter:

$$\beta = \max_{i=1,\ldots,n} \frac{x_i^*}{\sum_{j=1}^n x_j^*}. \tag{1}$$

If $\beta = 1$ then all items are sold to one single player. In this case, one bidder stands out, and the monopoly prefers to focus on him and extract all his surplus by setting a high price. Thus it is intuitively clear that the clinching auction cannot hope to extract a large fraction of the monopoly's revenue since there is no real competition. As β decreases, this "best" bidder faces more competition, and the clinching auction raises a larger fraction of the monopoly's revenue. Formally, we show:

Theorem 6.1. *The revenue of the clinching auction is at least a fraction of* $\frac{m}{m+n} \cdot (1 - \beta)$ *of a fixed-price monopoly's optimal revenue.*

Thus, the clinching auction revenue approaches the optimal monopoly revenue as indivisibility problems become less important and as bidder dominance decreases. The indivisibility issue, i.e., the fact that the bound in the theorem is interesting only when the number of items m is much larger than the number of bidders n, is a consequence of choosing to compare to a monopoly that can decide on fractional allocations of items to buyers. For example, suppose there is one item ($m = 1$) and every bidder $i = 1, \ldots, n$ has $v_i = n$ and $b_i = 1$. The monopoly will choose $p^* = 1$ and $x_i^* = 1/n$ for every bidder $i = 1, \ldots, n$, yielding a revenue of n, while the clinching auction has revenue 1 because it has to give the item integrally to one of the players. By this example, the bound in the theorem cannot be significantly improved.

Alternatively, we can compare to a monopoly that is also restricted to assign the items integrally. In this case, an alternative statement is that the revenue of the clinching auction is at least a fraction of $\frac{m}{2(m+\min(n,m))} \cdot (1 - \beta)$ of the monopoly's revenue. This claim follows using virtually the same proof we describe below (instead of Claim 6.2 we need to argue that without loss of generality the monopoly allocates at least half of the items).

We note that it is necessary to have some integrality factor even when comparing to a monopoly that assigns the items integrally. To demonstrate this, consider the following example. Suppose the number of items and bidders is equal, and all bidders have a budget 1 and value ∞. The monopoly sells one item to each player for a price of 1. The clinching auction sells one item to each player, for a price of 1/2, since at this price $D_i^+(1/2) = 1$ for every player i. Thus, there is a ratio of 1/2 between the revenue of the clinching auction and the monopoly's revenue.

[13] The argument is based on the following claim: if in the competitive equilibrium there is more than a single winner, then the revenue of this outcome is at least half of the optimal revenue (the maximal payment that satisfies individual rationality: $p_i \leqslant b_i$ and $p_i \leqslant x_i \cdot v_i$). Let us sketch the proof of this. Let p be the equilibrium price. Split the bidders to those with $v_i > p$ and those with $v_i \leqslant p$. The equilibrium revenue is $m \cdot p$. All bidders in the first set pay their full budget anyway in the equilibrium. We can never get more than a total of payment $m \cdot p$ from all bidders in the second set (since $v_i \leqslant p$). Thus the optimal revenue is at most $2m \cdot p$.

[14] In the classic definition for a (non-discriminatory) monopoly's revenue, one optimizes over all prices, where for each price the chosen allocation is the maximal individually rational allocation (i.e., each buyer consumes until the point where his marginal value equals the pre-determined unit price). We have reversed the optimization order (first over all allocations, and for each allocation over all individually rational prices), but the end outcome of the two optimization procedures is clearly identical. We choose the "reverse" description simply because it fits better with our proofs.

Proof of Theorem 6.1. We denote the optimal monopoly price by p^*, and the fractional assignment that maximizes the optimal monopolist price by $x^* = (x_1^*, \ldots, x_n^*)$. We show two claims:

Claim 6.2. *It can be assumed without loss of generality that all items are allocated in the fractional assignment that maximizes the optimal monopolist price. That is,* $\sum_i x_i^* = m$.

Proof. Assume that $\sum_i x_i^* < m$. Let $W = \{i \mid v_i \geqslant p^*\}$ and $B = \sum_{i \in W} b_i$. Since the unit-price is p^*, any player i with $v_i < p^*$ must have $x_i^* = 0$, hence the optimal monopoly price is at most B. Additionally, for any $i \in W$ we must have $x_i^* = b_i/p^*$ since otherwise we can increase the quantity that i gets, contradicting the fact that x^* maximizes the revenue. This implies that $\sum_{i \in W} b_i/p^* = \sum_{i \in W} x_i^* < m$, hence $p^* > B/m$. Now, by setting $p = B/m$ and $x_i = b_i/p$ for any $i \in W$ (note that $v_i \geqslant p^* > B/m = p$), we get revenue exactly B, and $\sum_i x_i = m$, thus the claim follows. □

Claim 6.3. *No player clinches an item before the price reaches* $\tilde{p} = \frac{m}{m+n} \cdot (1-\beta) \cdot p^*$.

Proof. We will show that, for each player i, $\sum_{j \neq i} D_j(\tilde{p}) \geqslant m$, which implies the claim. Let $W = \{j \mid x_j^* > 0\}$, and $W_{-i} = W \setminus \{i\}$. For any $j \in W$, $v_j \geqslant p^* > \tilde{p}$, hence $D_j(\tilde{p}) = \lfloor \frac{b_j}{\tilde{p}} \rfloor$. We therefore have

$$\sum_{j \neq i} D_j(\tilde{p}) \geqslant \sum_{j \in W_{-i}} D_j(\tilde{p}) = \sum_{j \in W_{-i}} \left\lfloor \frac{b_j}{\tilde{p}} \right\rfloor \geqslant \sum_{j \in W_{-i}} \left(\frac{b_j}{\tilde{p}} - 1 \right) \geqslant \sum_{j \in W_{-i}} \frac{b_j}{\tilde{p}} - n.$$

We next note that $\sum_{j \in W_{-i}} x_j^* = m - x_i^* \geqslant m - \beta \cdot m = m(1-\beta)$. This gives us:

$$\begin{aligned}
\sum_{j \neq i} D_j(\tilde{p}) &\geqslant \sum_{j \in W_{-i}} \frac{b_j}{\tilde{p}} - n = \frac{m+n}{m} \cdot \frac{1}{1-\beta} \cdot \sum_{j \in W_{-i}} \frac{b_j}{p^*} - n \\
&\geqslant \frac{m+n}{m} \cdot \frac{1}{1-\beta} \cdot \sum_{j \in W_{-i}} x_j^* - n \geqslant \frac{m+n}{m} \cdot \frac{1}{1-\beta} \cdot m(1-\beta) - n = m,
\end{aligned}$$

which proves the claim. □

We now prove Theorem 6.1. By Claim 6.2 we may assume that the optimal revenue is achieved by allocating all items and thus the optimal monopoly revenue is at most $m \cdot p^*$. The clinching auction sells all items (by Claim 3.1), and by Claim 6.3 each item is sold for a price of at least $\tilde{p} = \frac{m}{m+n} \cdot (1-\beta) \cdot p^*$. Thus the revenue of the clinching auction is at least $m \cdot \tilde{p} = \frac{m}{m+n} \cdot (1-\beta) \cdot (m \cdot p^*)$. □

7. The infinitely divisible good setting

While the clinching auction may be applied in the infinitely divisible setting by treating it as a continuous time process, the analysis for the divisible case is not a straight-forward extension of the indivisible case, and needs to rely on techniques different than the ones used in the discrete case. In this section we show explicit results for the divisible case that parallel our results for the indivisible case. We start by constructing a mechanism for known budgets in Section 7.1, exactly like our starting point in the indivisible case (the clinching auction in Section 3). In fact, there is a very close connection between our results in this section and the clinching auction: informally, we rely on the continuous time process that the clinching auction exhibits in the divisible case to obtain ("guess") an explicit closed-form auction for a divisible good and two players. We then directly prove that this closed-form mechanism is incentive compatible and Pareto optimal. We then show (Section 7.2) uniqueness of this mechanism among all anonymous mechanisms, assuming equal and public budgets (which is conceptually similar to the second step for the indivisible case, namely the uniqueness result in Section 4). Finally, Section 7.3 shows an impossibility result for anonymous mechanisms with private budgets, relying on the uniqueness result, in a similar manner to the way that the impossibility result for the indivisible case in Section 5 relies on the uniqueness result for that case.

7.1. A mechanism for known budgets

We construct a incentive compatible and Pareto optimal mechanism for two bidders with publicly known budgets. We start by analyzing two special cases, that will be used later on as building blocks for the general mechanism.

First special case: only one bidder with a budget limit. We first look at the case where only one of the players is budget-limited. Assume that $b_1 = 1$ (this is w.l.o.g.) and $b_2 = \infty$. There is an available quantity Q of a divisible item. Let us overview the course of the clinching auction in this case. Player 1 demands the entire quantity if the following two conditions hold:

(i) the price p is lower than his value v_1, and (ii) $Q \leqslant D_1(p) = \frac{b_1}{p} = \frac{1}{p}$. Player 2 demands the entire quantity if $p \leqslant v_2$ (since he is not constrained by a budget). Thus initially and as long as the price p is below both $1/Q$ and $\min(v_1, v_2)$, both players demand all the quantity, and so no clinching occurs. If $\min(v_1, v_2) \leqslant 1/Q$, the player i with the minimal value will drop out when the price will reach his value, and the other player will get the entire quantity and will pay the lower value.

Otherwise assume that $\min(v_1, v_2) > 1/Q$. When the price exceeds $1/Q$, player 1 starts reducing his demand to quantities smaller than Q. Therefore player 2 starts clinching the quantity that is not being demanded anymore by player 1. The total quantity clinched up to price p is $1 - D_1(p) = 1 - 1/p$ and thus player 2 clinches $d(1 - D_1(p))/dp = 1/p^2$ units at marginal price p. The total payment of player 2 up to some price x is therefore $\int_{1/Q}^{x} \frac{1}{p^2} p\, dp = \ln x - \ln(1/Q)$. This continues until the price reaches $\min(v_1, v_2)$ (recall that player 2 has infinite budget, hence he never reduces his demand unless the price reaches his value). Once we reach the point $p = \min(v_1, v_2)$, the low-value player drops, and the high-value player gets the remaining quantity at the current unit-price.

For example, if $Q = 1$, $\min(v_1, v_2) > 1$, and $v_2 > v_1$, player 2 receives the entire quantity and pays $\ln v_1$ (the payment until the point player 1 quits) plus $\frac{1}{v_1} v_1 = 1$ (the remaining quantity when player 1 quits is $\frac{1}{v_1}$ and this is sold to player 2 for a unit price v_1). This leads us to "guess" the following mechanism for this special case, and assuming $Q = 1$ ($Q = 1$ turns out to be all we need from this special case when constructing the general mechanism):

Definition 7.1 *(Mechanism M1 (assumes $Q = 1$)).*

- If $\min(v_1, v_2) \leqslant 1$ then the high-value player gets everything at the second price: $x_i = 1, p_i = v_j$ (and $x_j = 0, p_j = 0$), where $v_i > v_j$.
- Otherwise, if $v_2 \geqslant v_1$ then the high non-budget-limited player gets everything $x_2 = 1$ and pays $1 + \ln v_1$.
- Otherwise, if $v_1 > v_2$ then the high-value player gets $x_1 = 1/v_2$ and pays $p_1 = 1$, while the non-budget-limited player gets $x_2 = 1 - 1/v_2$ and pays $p_2 = \ln v_2$.

We give an explicit proof that mechanism M1 indeed satisfies Pareto optimality and incentive compatibility. In the proof, we use a slightly weaker assumption instead of $b_2 = \infty$, a relaxation that will become important in the sequel.

Proposition 7.2. *Fix any two budgets $b_1 \leqslant b_2$. Then, mechanism M1 is Pareto optimal and individually rational, and,*

1. *It is a dominant-strategy for player 1 to declare his true value.*
2. *If $v_2 \leqslant e^{b_2 - 1}$ then it is a dominant-strategy for player 2 to declare his true value. More precisely, let $u_2(z)$ denote player 2's resulting utility when he declares z. Then $u_2(v_2) \geqslant u_2(z)$ for any real number z.*

Proof. Pareto optimality follows directly from Proposition 2.4 since in the first two cases the low bidder gets allocated 0, and in the last case, the high bidder has his budget exhausted.

Let us start by looking at the incentives of bidder 1. If $v_2 \leqslant 1$ then he is faced with exactly two possibilities $x_1 = 1$, $p_1 = v_2$ and $x_1 = 0$, $p_1 = 0$. It is clear that he prefers the former if and only if $v_1 \geqslant v_2$, which is what happens with the truth. If $v_2 > 1$ then he is faced with two possibilities: either declare some $z \leqslant v_2$ in which case he gets $x_1 = 0$, $p_1 = 0$ or declare some $z > v_2$ and get allocated $x_1 = 1/v_2$, $p_1 = 1$. His utility in the first case is $u_i = 0$ and in the second $u_i = v_1/v_2 - 1$, which is positive iff $v_1 > v_2$ and given to him by the mechanism when telling the truth $z = v_1$.

Now for bidder 2. The case $v_1 \leqslant 1$ is as before. Otherwise he may declare either $z < v_1$ getting $x_2 = 1 - 1/z$, $p_2 = \ln z$ or declaring $z \geqslant v_1$ getting $x_2 = 1$, $p_2 = 1 + \ln v_1$. In the first case his utility is at most $u_2(z) = v_2 - v_2/z - \ln z$ (it is exactly this term if $p_2 \leqslant b_2$, otherwise it is smaller). This term for $u_2(z)$ is maximized for $z = v_2$ (by solving for $du_2/dz = 0$). Thus in the first case his utility is at most $v_2 - 1 - \ln v_2$. In the second case his utility at most $u_2 = v_2 - 1 - \ln v_1$. If $v_2 < v_1$ then the former term is larger than the latter term, and indeed by declaring $z = v_2$ the player obtains a utility exactly equal to $v_2 - 1 - \ln v_2$ since when $z = v_2$ we have $p_2 = \ln v_2 < \ln e^{b_2 - 1} < b_2$. If $v_2 \geqslant v_1$ then the latter term is better, and indeed by declaring $z = v_2$ the player obtains a utility exactly equal to $v_2 - 1 - \ln v_1$ since in this case $p_2 = 1 + \ln v_1 \leqslant 1 + \ln v_2 \leqslant 1 + \ln e^{b_2 - 1} = b_2$. Thus declaring $z = v_2$ obtains maximal utility, no matter what is v_1.

Individual rationality follows from incentive compatibility, since a player can always obtain a zero utility by declaring $v_i = 0$. □

Corollary 7.3. *Mechanism M1 is Pareto optimal and incentive compatible, assuming only one bidder is budget-constrained.*

Second special case: bidders with equal budgets. The second special case we analyze is when the budgets are equal and the available quantity is $Q \leqslant 1$ (here it will not be sufficient to assume $Q = 1$). Assume without loss of generality that $b_1 = b_2 = 1$ and $v_1 \leqslant v_2$.

We again "guess" a mechanism by looking at the course of the clinching auction. Similarly to before, while $p \leqslant 1/Q$ no clinching occurs since each player demands all available quantity. At the point $p = 1/Q$, the demand of both players is equal to available quantity, and hence from this point on both players will start clinching. Calculating the exact rate at which the clinching occurs is slightly more involved in this case. Let $D_i(p)$, $b_i(p)$ denote the current demand and remaining budget of player i at price p, and let $q_i(p)$ denote the total quantity that player i have received up to price p. When the price reaches $\min(v_1, v_2)$, the lower player drops and the high-value player receives the remaining quantity, but before this point the two players are completely identical, so we can remove the subscript i from the three functions. We have

$$D(p) = \frac{b(p)}{p}, \qquad b'(p) = -q'(p) \cdot p$$

where the second equation follows since budget at price p decreases by the quantity clinched at price p times the unit price paid for this clinched quantity. It will turn out useful to construct the three functions so that clinching will continuously occur, for all prices $p \geqslant 1/Q$. For this to happen, we need that the current demand of each player will always be exactly equal to the current available quantity (since in such a case, and only in such a case, when a player decreases her demand, the other player performs clinching). This means:

$$D(p) = Q - 2 \cdot q(p).$$

Solving these three equations, we get:

$$q(p) = \frac{Q}{2} - \frac{1}{2 \cdot Q \cdot p^2}, \qquad b(p) = \frac{1}{Q \cdot p}.$$

We next show explicitly that using these functions will indeed yield Pareto optimality and incentive compatibility. Moreover, in the sequel (Theorem 7.9) we show that this is the *unique* anonymous mechanism that is Pareto optimal and incentive compatible.

Definition 7.4 *(Mechanism M2 (for any initial quantity Q)).* Assume that $b_1 = b_2 = 1$ and $v_1 \leqslant v_2$.

- If $v_1 \leqslant 1/Q$ then the high-value player gets everything at the second price: $x_2 = Q$, $p_2 = v_1 \cdot Q$ (and $x_1 = 0$, $p_1 = 0$).
- Otherwise, the low-value player gets $x_1 = Q/2 - 1/(2 \cdot Q \cdot v_1^2)$ and pays $p_1 = 1 - 1/(Q \cdot v_1)$ and the high-value player gets $x_2 = Q/2 + 1/(2 \cdot Q \cdot v_1^2)$ and pays $p_2 = 1$.

Proposition 7.5. *Mechanism M2 is Pareto optimal, individually rational, and incentive compatible, in the case of publicly known and equal budgets.*

Proof. Pareto optimality follows directly from Proposition 2.4: in the first case the high-value player gets all the quantity, and in the second case the budget of the high-value player is exhausted.

Let us consider the incentives of one bidder with value v_i when the other bids a fixed value v_j. If $v_j \leqslant 1/Q$ then bidder i can choose between declaring $z \leqslant v_j$ in which case $x_i = 0$, $p_i = 0$ and thus $u_i = 0$ (in case of tie, if $x_i = 1$, $p_i = v_j$ then we still have $u_i = 0$) to bidding $z > v_j$ in which case $x_i = Q$, $p_i = v_j \cdot Q$ and thus $u_i = (v_i - v_j)Q$. The latter is better if and only if $v_i > v_j$, and by bidding $z = v_i$ player i gets the better option.

If $v_j > 1/Q$, then bidder i can choose between declaring $z < v_j$ in which case $x_i = Q/2 - 1/(2 \cdot Q \cdot z^2)$, $p_i = 1 - 1/(Q \cdot z)$ to bidding $z > v_j$ in which case $x_i = Q/2 + 1/(2 \cdot Q \cdot v_j^2)$, $p_i = 1$. Thus the utility when bidding $z < v_j$ is $v_i(Q/2 - 1/(2 \cdot Q \cdot z^2)) - (1 - 1/(Q \cdot z))$, and this is maximized by $z = v_i$. Thus the utility when bidding $z < v_j$ is at most $v_i(Q/2 - 1/(2 \cdot Q \cdot v_i^2)) - (1 - 1/(Q \cdot v_i))$ (call this $u^{(L)}$), and the utility when bidding $z > v_j$ is exactly $v_i(Q/2 + 1/(2 \cdot Q \cdot v_j^2)) - 1$ (call this $u^{(H)}$).

A short calculation shows that $u^{(L)} > u^{(H)}$ if and only if $v_i < v_j$. Therefore: (1) if $v_i < v_j$ then a player will maximize his utility by obtaining a utility equal to $u^{(L)}$, which can be obtained by declaring $z = v_i$, and (2) if $v_i > v_j$ then a player will maximize his utility by obtaining a utility equal to $u^{(H)}$, which can be obtained by declaring $z = v_i$. Thus no matter what is v_j, declaring v_i will maximize player i's utility. This proves incentive compatibility.

Individual rationality follows from incentive compatibility, since a player can always obtain a zero utility by declaring $v_i = 0$. □

The general case: bidders with arbitrary budgets. We now reach the case of general budgets, and we assume without loss of generality that $Q = 1$. While the general mechanism we define below does not seem related to the clinching auction in any apparent way, in fact it can be thought of as the "limit" of the clinching auction of Section 3, when the divisibility of the items increases, as we next explain. Assume that $b_1 = 1 < b_2$. As discussed above, if $p \leqslant v_i$ and $Q = 1 \leqslant D_i(p) = \frac{1}{p} \leqslant \frac{b_i}{p}$, both players demand all available quantity. Thus initially and as long as the price p is below both 1 and $\min(v_1, v_2)$, no clinching occurs since both players demand the entire quantity. If $\min(v_1, v_2) \leqslant 1$, the player i with the minimal value will drop out when the price will reach his value, and the other player will get the entire quantity and will pay the lower value.

The interesting situation is when $\min(v_1, v_2) > 1$. In this case by the above discussion no clinching will occur until the price will cross the point $p = 1$. At this point (and just after it) the course of the auction is similar to the first special case from above: player 2 still demands all quantity so player 1 does not perform clinching, and player 1 starts reducing his demand (as $b_1 = 1$), so player 2 starts to clinch. Using the equations found in the first special case above, the total clinched quantity of player 2 at price p is $q_2(p) = 1 - 1/p$, and his remaining budget is $b_2(p) = b_2 - \ln p$. This continues until the available quantity at price p equals the demand of player 2 at that price. This p point satisfies

$$\frac{b_2 - \ln p}{p} = \frac{b_2(p)}{p} = D_2(p) = 1 - q_2(p) = \frac{1}{p}$$

and the solution is $p^* = e^{b_2 - 1}$. To verify, note that at this price the available quantity is $1/p^*$, and the remaining budget of player 2 is $b_2(p^*) = 1$. Hence player 2 demands exactly the remaining quantity. Looking at player 1 we can see that, since he did not clinch anything up to p^*, his remaining budget is equal to his original budget, which was $b_1 = 1$. Thus the demand of player 1 at p^* is also $1/p^*$, again exactly equal to the remaining quantity. Therefore at p^* we have switched to a situation very similar to the second special case from above: both players have remaining budgets that are equal to 1, and at an initial price p^* they simultaneously demand exactly the available quantity. Thus, the calculations of the second special case of above, setting $Q = 1/p^*$, describe the course of the auction from this point until the end. In other words, we see that the general construction is simply a combination of the two special cases studied above. Note that the course of the above auction stops whenever the price reaches the point $\min(v_1, v_2)$, and this can be in any of the three parts of the auction – at $p < 1$, at $1 < p \leqslant p^*$, or at $p > p^*$. This description yields the intuition behind the general mechanism:

Definition 7.6 *(General mechanism).* Assume $b_1 = 1 \leqslant b_2$ and initial quantity of 1. Let $p^* = e^{b_2 - 1}$.

- If $\min(v_1, v_2) < p^*$ then run mechanism M1.
- Otherwise, allocate to player 2 an initial quantity of $1 - 1/p^*$ for a total price $b_2 - 1$. Allocate the remaining quantity $Q = 1/p^*$ using mechanism M2, where the initial budget of player 2 at the mechanism is $b_2 = 1$, and the rest of the parameters are unchanged.

Proposition 7.7. *The general mechanism is Pareto optimal and incentive compatible in the case of publicly known budgets.*

Proof. We first prove Pareto optimality. If $\min(v_1, v_2) < p^*$ then the outcome is determined by mechanism M1, hence is Pareto optimal by Proposition 7.2. If $\min(v_1, v_2) \geqslant p^*$, then mechanism M2 is run, and inside it we always enter the second option, which implies that the high-value player pays 1. If this is player 1 then this exhausts his budget, and if this is player 2 then his total payment is $(b_2 - 1) + 1 = b_2$, so his budget exhausted as well. Thus by Proposition 2.4 the outcome is indeed Pareto optimal.

We now prove incentive compatibility. Consider first the incentives of player 1. If $v_2 < p^*$ then mechanism M1 is used, no matter what player 1 reports, and the claim follows from Proposition 7.2. Otherwise $v_2 > p^*$. If $v_1 < p^*$ then by the properties of mechanism M2 player 1 prefers receiving zero utility to receiving some quantity as a result of declaring some $z > p^*$, since, in mechanism M2, when $v_1 < p^*$ player 1 gets nothing. Thus in this case player 1 maximizes utility by the incentive compatible declaration. If $v_1 > p^*$ then if he declares some $z < p^*$ he gets zero utility while if he declares v_1 he gets a non-negative utility since mechanism M2 is individually rational. Thus he prefers to declare some $z > p^*$ and since mechanism M2 is incentive compatible it must be that $z = v_1$. This proves incentive compatibility for player 1.

Now consider player 2. If $v_1 < p^*$ then the proof is a before. Otherwise $v_1 > p^*$. If $v_2 < p^*$ then player 2 prefers getting nothing from mechanism B to getting some positive quantity as a result of declaring some $z > p^*$, and he prefers getting from mechanism M1 a quantity that results from declaring v_2 to getting $1 - 1/p^*$ and paying $b_2 - 1$ (which results from declaring $z = p^*$). Thus player 2 prefers to declare v_2 over declaring some $z > p^*$, and therefore by the incentive compatibility of mechanism M1 he prefers to declare v_2 over any other declaration z. If $v_2 > p^*$ then player 2 prefers getting some quantity from mechanism M2 according to the declaration $z = v_2$ over not getting anything from mechanism M2, since mechanism M2 is individually rational. Additionally, player 2 prefers the outcome $x_2 = 1 - 1/p^*$, $p_2 = b_2 - 1$ over any other outcome that results from mechanism M1 by declaring some $z < p^*$, since $v_2(1 - 1/p^*) - \ln p^* > v_2(1 - 1/z) - \ln z$. Thus player 2 prefers the outcome resulting from declaring v_2 over any other outcome that results from declaring some $z < p^*$. By the incentive compatibility of mechanism M2, declaring v_2 will maximize player 2's utility. Therefore incentive compatibility for player 2 follows.

Individual rationality follows from incentive compatibility, since a player can always obtain a zero utility by declaring $v_i = 0$. □

7.2. Uniqueness for equal and known budgets

To show uniqueness we cannot simply use similar arguments to the ones of the discrete case, since there we used induction on the number of items, while here the number of items is fixed, in some sense. Thus we use completely different arguments, and rely on an additional property of anonymity: suppose that when player 1 declares v_1 and player 2 declares v_2, the outcome is that player i ($i = 1, 2$) gets x_i and pays p_i. Then, when player 1 declares v_2 and player 2 declares v_1, player 1 gets x_2 and pays p_2 and player 2 gets x_1 and pays p_1.

As defined, mechanism M2 is not really anonymous, breaking the tie $v_1 = v_2$ "in favor" of v_2. An anonymous mechanism with the same properties can be obtained by "splitting" in case of a tie:

Definition 7.8 *(Mechanism M3).*

- If $v_1 = v_2 = v \leqslant 1$ then $x_1 = x_2 = 1/2$ and $p_1 = p_2 = v/2$.
- If $v_1 = v_2 = v > 1$ then $x_1 = x_2 = 1/2$ and $p_1 = p_2 = 1 - 1/(2v)$.
- If $v_1 \neq v_2$ then run mechanism M2.

It is not hard to verify that mechanism M3 maintains incentive compatibility and Pareto optimality of mechanism M2. Moreover, we show:

Theorem 7.9. *Mechanism M3 is the only anonymous mechanism for the divisible good setting that satisfies individual rationality, incentive compatibility and Pareto optimality.*

Proof. Assume without loss of generality (as budgets are assumed to be equal) that $b_1 = b_2 = 1$. Let us fix a mechanism that satisfies the above properties and reason about it. In the rest of the proof we denote the smaller value by v_i, thus $v_i \leqslant v_j$.

Step 1: We first handle the case of $v_i \leqslant 1$. If also $v_j < 1$ then $p_j \leqslant v_j < 1$ and thus Pareto optimality implies $x_i = 0$ and $x_j = 1$. By the usual arguments of incentive compatibility we must have $p_j = v_i$. Now for values $v_j \geqslant 1$, if $x_j = 1$ then by incentive compatibility p_j is determined by x_j and thus is $p_j = v_i$. Otherwise $x_i > 0$ and thus by Pareto optimality $p_j = 1$ but this is a contradiction to incentive compatibility since declaring a value $v_i < v'_j < 1$ both increases x_j and decreases p_j.

Step 2: We will now show that there exist functions $q(t)$ and $p(t)$ such that whenever $v_i < v_j$ then $x_i = q(v_i)$, $p_i = p(v_i)$, and $x_j = 1 - q(v_i)$, $p_j = 1$. That is, the low player's value determines the allocation between the two players as well as his own payment, while the high-value player exhausts his budget. First assume to the contrary that for some $1 < v_i < v_j$, $p_j < 1$, and thus by Pareto optimality $x_i = 0$, $p_i = 0$, and $x_j = 1$. But then a bidder with $p_j < v'_j < 1 < v_i$ that, according to step 1, gets nothing, would be better off declaring v_j and getting positive utility, in contradiction to incentive compatibility. Thus $p_j = 1$ whenever $1 < v_i < v_j$. Thus, by incentive compatibility, for a fixed v_i, different values of v_j must get the same x_j, i.e. x_j depends only on v_i. By Pareto optimality, $x_i = 1 - x_j$ and thus it also only depends on v_i, and then by incentive compatibility p_i must be determined uniquely by x_i and thus depends only on v_i.

Step 3: Using incentive compatibility as usual, we have that for any $1 < t < t' < v_j$: $t(q(t') - q(t)) \leqslant p(t') - p(t) \leqslant t'(q(t') - q(t))$. As usual this implies that $dp/dt = t \cdot dq/dt$ or, more precisely, since we do not know that q is differentiable or even continuous, that $p(t) = tq(t) - \int_1^t q(x)\,dx$, where integrability of q is a direct corollary of its monotonicity.

Step 4: Using incentive compatibility and anonymity we have that for $1 < t < v_j < t'$: $tq(t) - p(t) \geqslant t(1 - q(v_j)) - 1$ since if player i has true value t it should not be beneficial for his to declare value t'. Similarly, $t'(1 - q(v_j)) - 1 \geqslant t'q(t) - p(t)$. Letting t, t' approach v_j we have that $v_j q(v_j) - p(v_j) = v_j(1 - q(v_j)) - 1$. Since this holds for every v_j we have $tq(t) - p(t) = t(1 - q(t)) - 1$, i.e. $p(t) = 1 + t(2q(t) - 1)$ for all t except for at most countably many points of discontinuity of q.

Step 5: Combining the last two steps we have $1 + t(2q(t) - 1) = tq(t) - \int_1^t q(x)\,dx$, i.e. $q(t) = 1 - 1/t - (\int_1^t q(x)\,dx)/t$, except for at most the countably many points of discontinuity of q. The solution to this differential equation is $q(t) = 1/2 - 1/(2t^2)$, which gives $p(t) = 1 - 1/t$. The uniqueness of solution is implied since if another function satisfies the equation everywhere except for countably many points, then the difference function $d(t)$ would satisfy $d(t) = -(\int_1^t d(x)\,dx)/t$ everywhere except for countably many points, which only holds for $d(t) = 0$. □

7.3. The impossibility for private budgets

From Theorem 7.9 we rather easily deduce:

Theorem 7.10. *There exists no anonymous, incentive compatible, and Pareto optimal mechanism for the divisible good setting, for the case of privately known budgets b_1, b_2.*

Proof. We first note that by direct scaling of Theorem 7.9 we have that the only mechanism that satisfies all requirements of the claim for the case of a publicly known budget $b_1 = b_2 = B$ gives $x_i = (1 - B^2/v_i^2)/2$, $p_i = B(1 - B/v_i)$, $x_j = (1 + B^2/v_i^2)/2$, $p_j = 1$ for the case $1 < v_i < v_j$, and $x_j = 1$, $p_j = v_i$, $x_i = 0$, $p_i = 0$ for the case $v_i < 1$ and $v_i < v_j$.

Let us now assume to the contrary that an auction that satisfies all requirements of the claim exists, then for any fixed values of b_1, b_2 it must be identical to the scaled version of mechanism M3. Now let us look at a few cases with $v_1 = 2$, $v_2 = 2 + \epsilon$. First let us look at the case $b_1 = b_2 = 1$. The previous theorem mandates that in this case $x_1 = 3/8$, $p_1 = 1/2$ and $x_2 = 5/8$, $p_2 = 1$ (and thus $u_2 = 1/4 + O(\epsilon)$).

Now let us look at the case where $b_1 = b_2 = 2 - \epsilon$. Again the Theorem 7.9 with scaling mandates that $x_1 > 0$ and also $u_1 > 0$.

Now let us look at the case of $b_1 = 1$ and $b_2 = 2 - \epsilon$. If $x_2 < 1$ then, by Pareto optimality, $p_2 = b_2 = 2 - \epsilon$, and thus $u_2 < 2\epsilon$, which means that player 2 has a profitable lie stating $b_2 = 1$. Thus $x_2 = 1$ and $x_1 = 0$, but then player 1 has a profitable lie stating that $b_1 = 2 - \epsilon$. □

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Appendix A. Proof of Claim 2.4

Recall that we need to show that an outcome $\{(x_i, p_i)\}$ is Pareto optimal in the infinitely divisible case if and only if (a) $\sum_i x_i = 1$ and (b) for all i such that $x_i > 0$ we have that for all j with $v_j > v_i$, $p_j = b_j$.

We first show that if either (a) or (b) do not hold then the outcome is not Pareto. If $\sum_i x_i < 1$ we simply add an additional quantity to some player for no additional charge, thus making him strictly better off while not harming any other player. Otherwise $\sum_i x_i = 1$ and there exists a player i with $x_i > 0$ and a player j with $v_j > v_i$ and $p_j < b_j$. Fix some ϵ such that $\epsilon \cdot v_i < b_j - p_j$. Construct an outcome (x', p') such that $x'_i = x_i - \epsilon$, $x'_j = x_j + \epsilon$, $p'_i = p_i - \epsilon \cdot v_i$, and $p'_j = p_j - \epsilon \cdot v_i$. All other players get the same quantity and pay the same price. Notice that $\sum_i p'_i = \sum_i p_i$ and that (x', p') is indeed a valid outcome. It is straight-forward to verify that i's utility remains the same while j's utility strictly increases.

For the other direction, fix an outcome (x, p) that satisfies (a) and (b). We will show that any other outcome (x', p') cannot be a Pareto improvement to (x, p) (as in Definition 2.3), implying that (x, p) is Pareto. Since (a) holds then $\sum_i x_i = 1$. Rename the players such that $v_1 \geqslant v_2 \geqslant \cdots \geqslant v_n$. Property (b) implies that there exists an index $1 \leqslant k \leqslant n$ such that, for any index $i < k$, $x_i > 0$ and $p_i = b_i$, for any index $i > k$, $x_i = 0$, and at k itself, $x_k > 0$. Let $\Delta = \sum_{i=1}^{k-1}(x_i - x'_i)$. For any i we need $u'_i \geqslant u_i$, which implies $p'_i - p_i \leqslant v_i \cdot (x'_i - x_i)$. We make several observations. First,

$$\sum_{i=k}^{n}(p'_i - p_i) \leqslant v_k(x'_k - x_k) + \sum_{i=k+1}^{n} v_i(x'_i - x_i) \leqslant v_k \sum_{i=k}^{n}(x'_i - x_i) = \Delta \cdot v_k$$

where the second inequality follows since $x_i = 0$ for any $i > k$, and the third inequality follows since $\sum_{i=1}^{k-1}(x_i - x'_i) - \sum_{i=k}^{n}(x'_i - x_i) = 0$. Second,

$$\sum_{i=1}^{k-1}(p_i - p'_i) \geqslant \sum_{1\leqslant i\leqslant k-1:\, x_i \geqslant x'_i}(p_i - p'_i) \geqslant \sum_{1\leqslant i\leqslant k-1:\, x_i \geqslant x'_i}(x_i - x'_i)v_i$$
$$\geqslant \sum_{1\leqslant i\leqslant k-1:\, x_i \geqslant x'_i}(x_i - x'_i)v_k \geqslant v_k \sum_{i=1}^{k-1}(x_i - x'_i) = \Delta \cdot v_k$$

where the first inequality follows since $p_i = b_i \geqslant p'_i$ for any $i < k$. Now, if there exists $1 \leqslant i \leqslant k-1$ such that $x_i < x'_i$ then the above argument yields $\sum_{i=1}^{k-1}(p_i - p'_i) > \Delta \cdot v_k$. We then get $\sum_{i=1}^{k-1}(p_i - p'_i) - \sum_{i=k}^{n}(p'_i - p_i) > 0$. In other words, $\sum_i p_i > \sum_i p'_i$, a contradiction to the definition of a Pareto improvement. Therefore assume that $x_i \geqslant x'_i$ for any $1 \leqslant i \leqslant k-1$. This implies that

$$\sum_{i=1}^{k-1}(x_i - x'_i)v_i \geqslant \Delta \cdot v_k \geqslant \sum_{i=k}^{n}(x'_i - x_i)v_i.$$

Putting together these four inequalities, we get

$$\sum_i(u_i - u'_i) = \sum_{i=1}^{k-1}(p_i - p'_i) - \sum_{i=k}^{n}(p'_i - p_i) + \sum_{i=1}^{k-1}(x_i - x'_i)v_i - \sum_{i=k}^{n}(x'_i - x_i)v_i \geqslant 0.$$

As a result, $u_i = u'_i$ for any player i, hence (x', p') is not a Pareto improvement for (x, p) since there does not exist a player i with $u'_i > u_i$. This concludes the proof of the claim.

Appendix B. Proof of Proposition 2.7

Let $\alpha_i = b_i / \sum_j b_j$. Recall that we need to prove that the proportional-share auction with $x_i = b_i / \sum_j b_j$ and $p_i = b_i$ is Pareto Optimal and incentive compatible in the range $v_i \geqslant \sum_j b_j / (1 - \alpha_i)$ for all i. Pareto optimality is trivial from Proposition 2.4 since we charge bidders their full budget. We now prove incentive compatibility in the specified range.

Since the values v_i do not affect the payment or the allocation, it suffices to show that no manipulation of b_i is profitable. Since we charge each bidder his total declared budget, it is clear that declaring $b'_i > b_i$ will lead to the bidder exceeding his budget. Thus it suffices to prove that no smaller declaration $b'_i < b_i$ is profitable. Let $u(z)$ be the utility obtained by bidder i if he declares a budget of $b'_i = z$. Thus $u(z) = v_i \cdot z/(z + \sum_{j\neq i} b_j) - z$. It suffices to show that u is monotonically increasing with z. To verify this, take the derivative with respect to z: $u'(z) = v_i \sum_{j\neq i} b_i/(z + \sum_{j\neq i} b_j)^2 - 1$. This derivative is non-negative, $u'(z) \geqslant 0$, if $v_i \geqslant (\sum_j b_j)^2 / \sum_{j\neq i} b_j = \sum_j b_j/(1-\alpha_i)$, as is indeed specified. This concludes the proof of the claim.

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