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Walrasian Equilibrium with Gross Substitutes*

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We study economies with indivisibilities that satisfy the gross substitutes (GS) condition. The simplest example of GS preferences are unit demand preferences. We prove that the set of GS preferences is the largest set containing unit demand preferences for which the existence of Walrasian equilibrium is guaranteed. We show that if a GS economy is replicated sufficiently many times, the equilibrium payment of any agent in the Vickrey–Clarke–Groves mechanism is equal to the value of the allocation he receives at the smallest Walrasian prices. The model extended to include production. *Journal of Economic Literature* Classification Numbers: D4, D44, D5, D51. © 1999 Academic Press

1. INTRODUCTION

In this paper we study the problem of efficient production and allocation of indivisible objects among a set of consumers. We assume that the agents' preferences depend on the bundle of objects and the quantity of money they consume. Furthermore, we assume that preferences are quasilinear in money, and that agents have a large endowment of money.

With indivisibilities, it is well-known that many familiar properties of the utility functions fail to ensure existence. In their striking analysis of the matching problem, Kelso and Crawford [5] introduce the gross substitutes (GS) condition which ensures the non-emptiness of the core. We propose two new conditions, and show that with quasilinearity they are equivalent

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to the GS condition of Kelso and Crawford. The simplest example of GS preferences are unit demand preferences. A unit demand preference is such that the agent can enjoy at most one object. We prove that the set of GS preferences is the largest set containing unit demand preferences for which an existence theorem can be established. Thus, we prove a "converse" to Kelso and Crawford's existence result; in a sense, the GS condition is necessary to ensure existence of a Walrasian equilibrium.

With quasilinear preferences, there is a representative consumer whose demand function coincides with society's aggregate demand. When the GS condition is satisfied, the smallest Walrasian price and the largest Walrasian price of each object can be interpreted as shadow prices. The largest Walrasian price of any object α represents the decrease in total utility of an efficient allocation that would result if this object were removed from the aggregate endowment. Similarly, the smallest Walrasian price represents the amount of increase associated with an efficient allocation if a second copy (i.e. a perfect substitute) of this object were added to the economy. Consequently, we show that the representative consumer's utility function satisfies submodularity whenever the utility functions of the individual agents satisfy the GS condition.

In Section 5 we compare Walrasian prices with Vickrey–Clarke–Groves payments. We prove that for any profile of preferences, the equilibrium payment of any agent in the VCG mechanism is less then or equal to the value of the allocation he receives at the smallest Walrasian prices. Therefore, the total revenue raised by the VCG mechanism is less than or equal to the value of the aggregate endowment at the smallest Walrasian prices. We show that these inequalities may be strict. However, the two inequalities are in fact equalities if the initial economy satisfies the GS condition and is replicated $m+1$ times (where m is the number of objects in the initial economy).

In Section 6 we generalize the model to include production.

Kelso and Crawford's analysis of the core of a matching problem plays a central role in our work. Their framework is more general than ours. In particular, they do not impose quasilinearity. We rely on their paper for existence of a competitive equilibrium and utilize quasilinearity to prove additional results. We compare our results to theirs throughout the paper.

A different approach to the existence problem is provided in Bikhchandani and Mamer [2]. They construct a related economy with quasilinear preferences without indivisibilities. The total surplus attainable in this economy is no less than the total surplus attainable in the economy with indivisibilities. Their main theorem proves that equilibrium in the economy with indivisibilities exists if and only if the maximal attainable surplus is equal to the maximal attainable surplus in the corresponding economy with no indivisibilities. They use this result to identify various sufficient conditions for existence with indivisibilities. The necessary and sufficient condition for their main theorem described above suggests the following alternative approach for proving existence of Walrasian equilibrium. One could verify directly that an economy satisfying the GS condition and its divisible analog defined in Bikhchandani and Mamer [2] yield the same surplus.

2. PREFERENCES

In this section we study properties of the consumers' preferences. We confine attention to preferences that are quasilinear in money (that is, to additively separable utility functions), and study conditions on the preferences over bundles of objects. $\Omega = \{\omega_1, ..., \omega_m\}$ is the set of objects in the economy. A *bundle* is any subset B of Ω; the set of all bundles is

$$2^{\Omega} := \{B \mid B \subset \Omega\}.$$

A price vector $p \in \mathbf{R}^m_+$ includes a price for each *object* in Ω.

DEFINITION. A map $u\colon 2^{\Omega} \to \mathbf{R}$ is called a *utility function* on Ω. A utility function assigns a value to each *bundle* of Ω. With each utility function u we associate the *net utility function* $v\colon 2^{\Omega} \times \mathbf{R}^m_+ \to \mathbf{R}$, which is defined by

$$v(A, p) := u(A) - \langle p, A \rangle, \qquad \text{where} \quad \langle p, A \rangle := \sum_{a \in A} p_a$$

(and by convention, $\langle p, \varnothing \rangle := 0$).

DEFINITION. A utility function $u\colon 2^{\Omega} \to \mathbf{R}$

(i) is *monotone* if for all $A \subset B \subset \Omega$, $u(A) \leqslant u(B)$.

(ii) is *submodular* if for all $A, B \subset \Omega$,

$$u(A) + u(B) \geqslant u(A \cup B) + u(A \cap B).$$

(iii) has *decreasing marginal returns* if for all $A \subset B \subset \Omega$ and $a \in A$,

$$u(B) - u(B \backslash \{a\}) \leqslant u(A) - u(A \backslash \{a\}).$$

If $u(\varnothing) = 0$ and u is monotone, then $u(A) \geqslant 0$ for all $A \subset \Omega$. In what follows, without loss of generality, we normalize every utility function u so that $u(\varnothing) = 0$.

Conditions (i)–(iii), as well as the equivalence of conditions (ii) and (iii), are well known in the literature. Likewise, one can establish the equivalence between supermodularity (or convexity), which is obtained by reversing the inequality in the definition of submodularity, and increasing marginal returns (which is obtained by reversing the inequality in the definition of decreasing marginal returns).

LEMMA 1. *u is submodular iff u has decreasing marginal returns.*

The proof of this lemma can be found, for example, in Moulin [10].

DEFINITION. For any utility function $u: 2^{\Omega} \to \mathbf{R}$, its *demand correspondence* $D: \mathbf{R}^m_+ \to 2^{\Omega}$ is defined by

$$D(p) := \{A \subset \Omega \mid v(A, p) \geqslant v(B, p) \text{ for all } B \subset \Omega\}, \qquad p \in \mathbf{R}^m_+.$$

DEFINITION. Let A, B, and C be any three bundles. Then $\#(A)$ denotes the number of elements in A,

$$A \Delta B := [A \backslash B] \cup [B \backslash A]$$

is the *symmetric difference* between A and B, $\#(A \Delta B)$ is the *Hausdorff distance* between A and B, and

$$[A, B, C] := (A \backslash B) \cup C.$$

If B is a singleton $\{b\}$, we write $[A, b, C]$ instead of $[A, \{b\}, C]$ (and similarly if C is a singleton).

It is easy to see that for any utility function $u: 2^{\Omega} \to \mathbf{R}$, its demand correspondence $D: \mathbf{R}^m_+ \to \Omega$ is upper semicontinuous when 2^{Ω} is endowed with the Hausdorff metric. That is, if $\{p_k\}$ is a sequence of price vectors converging to $\bar{p}$ and $A \in D(p_k)$ for all k, then $A \in D(\bar{p})$.

The following definition presents four closely related properties for a utility function: (GS), (SI), (NC), and (SNC). The first was originally introduced by Kelso and Crawford [5]; the other three are new.

DEFINITION. A utility function $u: 2^{\Omega} \to \mathbf{R}$

(i) satisfies the *gross substitutes* condition (GS) if for any two price vectors p and q such that $q \geqslant p$, and any $A \in D(p)$, there exists $B \in D(q)$ such that $\{a \in A \mid p_a = q_a\} \subset B$.

(ii) has the *single improvement* property (SI) if for any price vector p and bundle $A \notin D(p)$, there exists a bundle B such that $v(A, p) < v(B, p)$, $\#(A \backslash B) \leqslant 1$, and $\#(B \backslash A) \leqslant 1$.

(iii) has *no complementarities* (NC) if for each price vector p, and all bundles $A, B \in D(p)$ and $X \subset A$, there exists a bundle $Y \subset B$ such that $[A, X, Y] \in D(p)$.

(iv) satisfies, the *strong no complementarities* condition (SNC) if for all $A, B \subset \Omega$ and $X \subset A$, there exists $Y \subset B$ such that

$$u(A) + u(B) \leqslant u([A, X, Y]) + u([B, X, Y]).$$

Remark. To check whether u has no complementarities, it is enough to consider the cases in which $X \subset A \backslash B$. And for these cases, we only need to search among bundles $Y \subset B \backslash A$.

Suppose an agent with utility function u wants to consume a bundle A at prices p. Condition GS states that if the prices were increased from p to q, then the agent would still want to consume the objects in A whose prices did not increase. That is, at q there is an optimal bundle B that includes all those objects (and possibly others). Condition SI states that any suboptimal bundle A at prices p can be strictly improved by either removing an object from it, or adding an object to it, or doing both. Suppose A and B are two optimal bundles at prices p, and an arbitrary part X is removed from A. Condition NC says that a new optimal bundle can be constructed with the objects that are left and a part Y of the bundle B. Finally, condition SNC has the following interpretation. Suppose that two identical agents have utility function u, and are endowed with bundles A and B respectively (not necessarily disjoint). Suppose agent 1 hands agent 2 a subset X of her endowment. If u has no complementarities, agent 2 should be able to return to agent 1 a subset Y of his initial endowment, so that their total utility after the swap is preserved or increased.

The following piece of notation is used throughout the paper. In particular, it is used in the Appendix, where we present the proof of Theorem 1 divided into Lemmas 2–4, and in Section 4.

Notation. If A is a bundle, $e^A \in \mathbf{R}^m$ denotes its *characteristic vector*, whose coordinates are $e^A_a = 1$ for $a \in A$, and $e^A_a = 0$ otherwise. If A is a singleton $\{a\}$, we sometimes write e^a instead of e^A.

THEOREM 1. *If u is monotone, then GS, SI, and NC are equivalent.*

It is easy to verify that SNC implies NC, and therefore, by Theorem 1, GS and SI as well. While SNC is a stronger condition, it has the advantage of being stated directly in terms of the utility function rather than the demand correspondence. Kelso and Crawford [5] use GS to prove their main results. However, SI turns out to be more appropriate for our analysis (i.e., in establishing that the set of Walrasian equilibrium prices is a lattice).

LEMMA 5. *If u is monotone and satisfies GS, then u and $v(\cdot, p)$ are submodular for any price vector p.*

Proof. We first show that u is submodular. By Lemma 1, it is enough to show that u has decreasing marginal returns. Let $\alpha \in A \subset B \subset \Omega$. Define the price vector p as follows: $p_a = 0$ for all $a \in B$ and $p_a = M > u(\Omega)$ otherwise. By monotonicity, $B \in D(p)$. For each $\varepsilon \geqslant 0$, let $q(\varepsilon) := p + \varepsilon e^{\alpha}$, and define

$$\bar{\varepsilon} := \max\{\varepsilon \mid B \in D(q(\varepsilon))\}.$$

Since D is upper semicontinuous, $B \in D(q(\bar{\varepsilon}))$. By GS, for each $\varepsilon \geqslant 0$, there exists $C \in D(q(\varepsilon))$ such that $C \supset B \backslash \{\alpha\}$. Since for each $\varepsilon > \bar{\varepsilon}$, $B \notin D(q(\varepsilon))$, we must have that $B \backslash \{\alpha\} \in D(q(\varepsilon))$, and by upper semicontinuity, $B \backslash \{\alpha\} \in D(q(\bar{\varepsilon}))$. Therefore

$$u(B) - \bar{\varepsilon} = v(B, q(\bar{\varepsilon})) = v(B \backslash \{\alpha\}, q(\bar{\varepsilon})) = u(B \backslash \{\alpha\}). \tag{1}$$

Now, define the price vector r as follows: $r_\alpha := \bar{\varepsilon}$, $r_a := 0$ for all $a \in A \backslash \{\alpha\}$, and $r_a := M$ for all $a \notin A$. Clearly, if X is any bundle such that $X \not\subset A$, then $X \notin D(r)$. Since $A = \{a \in B \mid r_a = q_a(\bar{\varepsilon})\}$, GS implies that $A \in D(r)$. Therefore

$$u(A) - \bar{\varepsilon} = v(A, r) \geqslant v(A \backslash \{\alpha\}, r) = u(A \backslash \{\alpha\}). \tag{2}$$

Equations (1) and (2) imply that

$$u(B) - u(B \backslash \{\alpha\}) = \bar{\varepsilon} \leqslant u(A) - u(A \backslash \{\alpha\}).$$

Hence, u has decreasing marginal returns. Finally, $v(\cdot, p)$ is submodular because it is the sum of two submodular functions. ∎

Kelso and Crawford [5] provide an example showing that submodularity and monotonicity do not imply the GS condition. Thus, the converse of Lemma 5 is false.

DEFINITION. A utility function u represents a *unit demand preference* if $u(\varnothing) = 0$ and for each nonempty bundle A,

$$u(A) = \max_{a \in A} u(\{a\}).$$

A unit demand utility function u is completely specified by the values it assigns to singletons and the empty set, and we will sometimes abuse notation and write $u(a)$ instead of $u(\{a\})$, for $a \in \Omega$. Every unit demand utility function satisfies the SNC condition.

Koopmans and Beckmann [6] study exchange economies where all consumers have unit demand preferences. They show that the Walrasian equilibrium problem is equivalent to the standard (linear programming) assignment problem, and that Walrasian prices coincide with its dual variables. They note that the dual problem always has a solution, and thus establish the existence of Walrasian prices. In the same setting, Leonard [8] shows that Walrasian prices can be interpreted as marginal values of the (society's) surplus function, and discusses the incentive compatibility of a generalization of the Vickrey auction (see Section 5 below).

Two other classes of functions that satisfy the GS condition are the set of additively separable utility functions and the set of additively concave functions. An additively separable utility function u is also completely specified by the values it assigns to singletons. Its value for any bundle A is given by

$$u(A) = \sum_{a \in A} u(\{a\}).$$

An additively concave utility function partitions Ω into sets of "homogeneous" goods. Suppose that there are only two distinct objects α and β, and that Ω contains several units of each: $\Omega = \{\alpha_1, ..., \alpha_r, \beta_1, ..., \beta_s\}$. Let $\mathbf{N}$ denote the set of nonnegative integers. Assume that there are two increasing functions $u_\alpha, u_\beta \colon \mathbf{N} \to \mathbf{R}$ such that $u(A) = u_\alpha(x) + u_\beta(y)$ whenever A contains x units of α and y units of β. If u_α and u_β are "concave" (that is, have decreasing marginal returns), then u satisfies the SNC condition. Conversely, if u satisfies the GS condition, u_α and u_β must be concave.

Bevia, Quinzii and Silva [1] have introduced a class of preferences that can be represented by utility functions u satisfying the property

$$u(A) = \sum_{a \in A} u(\{a\}) - c(\#(A)) \qquad A \subset \Omega,$$

where $c \colon \mathbf{N} \to \mathbf{R}$. Any such a utility function satisfies the GS condition if c is "convex" (that is, has increasing marginal returns).

In addition to these classes of GS preferences, there are two operations that allow us to derive new GS preferences from other known GS preferences. Suppose that u_1 and u_2 are two GS functions on Ω and that there are two bundles A_1 and A_2 such that $A_1 \cap A_2 = \varnothing$ and $u_i(A_i) = u_i(\Omega)$, $i = 1, 2$. Then, the utility function u, defined by $u(B) = u_1(B) + u_2(B)$ for each $B \subset \Omega$, satisfies the GS condition. For any $k < m$, the k-satiation of any utility function u is the utility function $\hat{u}$ defined by

$$\begin{aligned} \hat{u}(A) = \max\ & u_i(B) \\ \text{s.t.}\quad & B \subset A \quad \text{and} \quad \#(B) \leqslant k. \end{aligned}$$

If u is additively separable or additively concave, $\hat{u}$ satisfies the GS condition. The k-satiation of an additively separable utility function results in a natural extension of a unit demand preference.

3. WALRASIAN EQUILIBRIA

The economy $E=(\Omega; u_1, ..., u_n)$ consists of the finite collection of objects Ω, and the set of consumers $N:=\{1, ..., n\}$. Each consumer i has a quasilinear utility function $U_i: 2^{\Omega} \times \mathbf{R} \to \mathbf{R}$; for each bundle $A \subset \Omega$ and money amount $t \in \mathbf{R}$ (she has for consumption of other goods), $U_i(A, t) = u_i(A) + t$, where $u_i: 2^{\Omega} \to \mathbf{R}$. Without loss of generality, we normalize so that $u_i(\emptyset)=0$, and assume that each consumer i is endowed with a sufficient amount of money $M_i > u_i(\Omega)$. We denote by v_i consumer i's corresponding net utility function. We also assume the economy has free disposal, and let $N_0 := N \cup \{0\}$.

Note that out description of the economy does not make any reference to endowments. Due to quasilinearity and our assumption that each agent is endowed with a large amount of money, the set of Walrasian equilibrium allocations of objects and the associated prices, as well as the set of efficient allocations of objects, are independent of the initial endowments of objects. Thus, we choose to ignore the initial endowment and characterize efficiency only in terms of the allocation of objects. Walrasian equilibria are fully described by the allocation of objects, the prices of the goods and the implied transfers of money.

Definition. $\mathbf{X}=(X_0, ..., X_n)$ is a *partition* (or *allocation*) of Ω if (1) $X_i \cap X_j = \emptyset$ for all $i \neq j$; and (2) $\bigcup_{i=0}^{n} X_i = \Omega$. The possibility that $X_i = \emptyset$ for some i is allowed. For $i \in N$, X_i represents consumer i's consumption bundle, and X_0 represents the collection of objects that are not consumed by anyone.

Definition. The tuple $(X_0, ..., X_n; t_1, ..., t_n)$, where (X_i, t_i) represents the bundle and money amount consumed by i, is an *outcome* for the economy if it satisfies the feasibility constraints: (1) $\sum_{i \in N} t_i = \sum_{i \in N} M_i$; and (2) $(X_0, ..., X_n)$ is a partition of Ω (since each object $\omega \in \Omega$ can be consumed by at most one consumer).

Definition. A *Walrasian equilibrium* for the economy $E=(\Omega; u_1, ..., u_n)$ is a tuple $(p, \mathbf{X})$, where $p \in \mathbf{R}_+^m$ is a price vector, and $\mathbf{X}=(X_0, ..., X_n)$ is a partition of Ω such that (1) $\langle p, X_0 \rangle = 0$, and (2) for each $i \in N$, $v_i(X_i, p) \geqslant v_i(A, p)$ for all bundle $A \subset \Omega$.

Let $(p, \mathbf{X})$ be a Walrasian equilibrium. Since $\langle p, X_0 \rangle = 0$, $p_a = 0$ for each $a \in X_0$, and if u_1 is monotone, $v_1(X_1 \cup X_0, p) \geqslant v_1(X_1, p)$. That is, at prices p, $X_1 \cup X_0$ is also an optimal bundle for consumer 1. Thus, $(p, \hat{\mathbf{X}})$, where $\hat{X}_0 = \emptyset$, $\hat{X}_1 = X_1 \cup X_0$, and $\hat{X}_j = X_j$ for each $j \geqslant 2$, is also a Walrasian equilibrium. Therefore, without loss of generality, we will sometimes assume that the Walrasian equilibria $(p, \mathbf{X})$ we choose satisfy the additional requirement that $X_0 = \emptyset$.

Existence of a Walrasian equilibrium in our model is implied by Theorem 3 in Kelso and Crawford [5], that guarantees the existence of strict core allocations. We restate in our notation the definition of a strict core allocation.

Definition. $(p, \mathbf{X})$, where $p \in \mathbf{R}^m_+$ and $\mathbf{X}$ is a partition of Ω, is a *strict core allocation* if $X_0 = \emptyset$ and there does not exist an agent i, a bundle Y_i, and a price vector $q \geqslant p$, such that $v_i(Y_i, q) > v_i(X_i, p)$.

Theorem 3 in Kelso and Crawford [5] shows that if agents' preferences satisfy the GS and the MP condition, then a strict core allocation for E exists. Condition MP is equivalent in our model to the monotonicity of the agents' utilities. Hence, if each agent's utility satisfies GS, E has a strict core allocation. It is easy to show that $(p, \mathbf{X})$ is a strict core allocation iff $(p, \mathbf{X})$ is a Walrasian equilibrium with $X_0 = \emptyset$. Hence, if all preferences are monotone and satisfies the GS condition, then a Walrasian equilibrium exists.

Theorem 2 below establishes that in a sense GS is a "necessary" condition for existence of Walrasian equilibrium. For any consumer with a monotone utility function u that fails the GS condition, one can find a collection of unit demand consumers such that the resulting economy has no Walrasian equilibrium. Thus, Theorem 2 shows that the Kelso and Crawford's existence theorem is the strongest possible generalization of Koopmans and Beckmann's result for unit demand economies. The proof of Theorem 2 is relegated to the Appendix.

Theorem 2. *Consider a consumer with a utility function $u_1: 2^{\Omega} \to \mathbf{R}$ that violates SI. Then, there exist $\ell - 1$ unit demand consumers with utility functions u_i, $i = 2, ..., \ell$, such that the economy $E = (\Omega; u_1, ..., u_\ell)$ does not have a Walrasian equilibrium.*

The standard theorems of welfare economics hold for our economy E. However, for several proofs below we need the following slightly stronger second theorem of welfare economics.

It is easy to see that an outcome $(\mathbf{X}, t)$ is Pareto efficient iff $\mathbf{X}$ maximizes total utility.

Lemma 6. *If p is any Walrasian price vector and* $\mathbf{Y}$ *is any efficient allocation, then* $(p, \mathbf{Y})$ *is a Walrasian equilibrium.*

Proof. Suppose the allocation $\mathbf{X}$ is such that $(p, \mathbf{X})$ is a Walrasian equilibrium. Then $\langle p, X_0 \rangle = 0$ and $v_i(X_i, p) \geqslant v_i(Y_i, p)$ for each $i \in N$. And since $\mathbf{Y}$ is efficient, we have

$$\sum_{i \in N} u_i(Y_i) - \langle p, \Omega \rangle \geqslant \sum_{i \in N} u_i(X_i) - \langle p, \Omega \rangle = \sum_{i \in N} v_i(X_i, p)$$

$$\geqslant \sum_{i \in N} v_i(Y_i, p) = \sum_{i \in N} u_i(Y_i) - \langle p, \Omega \rangle + \langle p, Y_0 \rangle.$$

These inequalities imply that $\langle p, Y_0 \rangle = 0$ and $v_i(Y_i, p) = v_i(X_i, p)$ for each $i \in N$ (that is, Y_i is also an optimal bundle for consumer i at prices p). ∎

Definition. Let p and q be two price vectors. Their *join* $r = p \vee q$ and *meet* $s = p \wedge q$ are the price vectors defined by

$$r_a := \max\{p_a, q_a\} \qquad \text{and} \qquad s_a := \min\{p_a, q_a\} \qquad \text{for each} \quad a \in \Omega.$$

A set of price vectors P is a *lattice* if for all $p, q \in P$, both $p \vee q \in P$ and $p \wedge q \in P$. The lattice P is *complete* if for any $Q \subset P$, $\bigwedge(Q) \in P$ and $\bigvee(Q) \in P$, where

$$\bigwedge(Q) := \inf\{q_a \mid q \in Q\} \quad \text{and} \quad \bigvee(Q) := \sup\{q_a \mid q \in Q\} \quad \text{for all } a \in \Omega.$$

Definition. A price vector p *supports* a partition $\mathbf{X}$ of Ω if $v_i(X_i, p) \geqslant v_i(A, p)$ for each bundle A and consumer i. A price vector *supports* a bundle A if p supports a partition $\mathbf{X}$ of Ω such that $X_0 = \Omega \backslash A$.

Observe that if p supports $\mathbf{X}$, then $(p, \mathbf{X})$ is a Walrasian equilibrium iff $\langle p, X_0 \rangle = 0$.

The next result, together with Theorems 4 and 5 of Section 4, enables us to interpret Walrasian prices as shadow prices.

Theorem 3. *Assume u_i has the SI property for each $i \in N$. Then, the set of prices that support a partition* $\mathbf{X}$ *of Ω is a complete lattice.*

Proof. Let P denote the set of all prices that support $\mathbf{X}$. If P is empty, we are done. Otherwise, let $p, q \in P$ and $r := p \wedge q$, and assume that r does not support the partition $\mathbf{X}$. Then there exists $i \in N$ and bundle Z such that $v_i(X_i, r) < v_i(Z, r)$. By SI, we can assume that $Z = [X_i, A, B]$, where $A \subset X_i$, $B \cap X_i = \varnothing$, A is either empty or a singleton $\{a\}$, and B is either empty or a singleton $\{b\}$. The inequality $v_i(X_i, r) < v_i(Z, r)$ is equivalent to

$$u_i(Z) - u_i(X_i) > \langle r, Z \rangle - \langle r, X_i \rangle = \langle r, B \rangle - \langle r, A \rangle. \tag{$*$}$$

If $\langle r, A\rangle = \langle p, A\rangle \leqslant \langle q, A\rangle$ and $\langle r, B\rangle = \langle p, B\rangle \leqslant \langle q, B\rangle$, the above inequality implies that $v_i(Z, p) > v_i(X_i, p)$, contradicting the fact that p supports X. Similarly, if $\langle p, A\rangle \geqslant \langle q, A\rangle = \langle r, A\rangle$ and $\langle p, B\rangle \geqslant \langle q, B\rangle = \langle r, B\rangle$, $v_i(Z, q) > v_i(X_i, q)$, another contradiction. Therefore, both A and B are nonempty and either $[r_a = p_a < q_a$ and $r_b = q_b < p_b]$ or $[r_a = q_a < p_a$ and $r_b = p_b < q_b]$. Assume the former. Then

$$\langle r, B\rangle - \langle r, A\rangle = q_b - p_a > q_b - q_a,$$

and $(*)$ above implies that $v_i(Z, q) > v_i(X_i, q)$, a contradiction. By symmetry, if we assume the latter, we obtain $v_i(Z, p) > v_i(X_i, p)$, another contradiction. Therefore $r = p \wedge q$ supports $\mathbf{X}$.

The proof that $p \vee q$ also supports $\mathbf{X}$ follows a similar argument, and is omitted. Hence P is a lattice.

For each $a \in \Omega$, the projection function $\varphi(p) := p_a$, $p \in \mathbf{R}^m_+$, is continuous. Therefore, to prove that P is complete, it is enough to show that P is closed. But $p \in P$ iff it satisfies the linear constraints

$$\langle p, X_i\rangle - \langle p, A\rangle \leqslant u_i(X_i) - u_i(A) \qquad \text{for all} \quad i \in N \quad \text{and} \quad A \subset \Omega.$$

That is, P is a closed simplex in $\mathbf{R}^m_+$. ∎

Corollary 1. *Suppose u_i has the SI property for each $i \in N$. Then, the set P^W of Walrasian equilibrium prices is a complete lattice.*

Proof. By Kelso and Crawford [5], P^W is nonempty. Suppose $p, q \in P^W$ and $\mathbf{X}$ is an efficient allocation. Then, by Lemma 6, $(p, \mathbf{X})$ and $(q, \mathbf{X})$ are Walrasian equilibria, and both p and q support $\mathbf{X}$. By the previous Theorem, $p \vee q$ and $p \wedge q$ also support $\mathbf{X}$. Since $\langle p, X_0\rangle = \langle q, X_0\rangle = 0$, we have that $\langle p \vee q, X_0\rangle = \langle p \wedge q, X_0\rangle = 0$. Therefore $(p \vee q, \mathbf{X})$ and $(p \wedge q, \mathbf{X})$ are Walrasian equilibria, and P^W is a lattice. Finally, for *any* efficient partition $\mathbf{X}$, P^W is equal to the set of prices p that support $\mathbf{X}$ and satisfy the additional linear constraint $\langle p, X_0\rangle = 0$. Therefore, P^W is a closed simplex, and thus it is a complete lattice as well. ∎

Definition. Let $\underline{p} := \bigwedge(P^W)$ and $\bar{p} := \bigvee(P^W)$, where P^W is the set of Walrasian prices for E.

By Corollary 1 and the existence of Walrasian equilibria, $\underline{p}$ and $\bar{p}$ exist and are themselves Walrasian prices.

With two additional conditions, NTW and NTF, which are "generically" satisfied, Kelso and Crawford [5] establish that their discrete salary adjustment process converges to the best discrete core allocation for the agents (see their Theorem 4). In our context, the best discrete core allocation for the consumers is equivalent to the smallest Walrasian prices. However, the NTW and NTF conditions have no counterparts in our model since we do not consider discretized economies (i.e., a smallest unit of currency).

4. SOCIAL SURPLUS

In this section we establish that society's largest and smallest marginal valuations of an object coincide with the object's largest and smallest Walrasian prices respectively.

It is necessary here to study situations that involve allocations that are infeasible. That is, we need to allow for allocations $\mathbf{X} = (X_1, ..., X_n) \subset \Omega^n$ with bundles that are not necessarily pairwise disjoint. Alternatively, we can interpret such an allocation as if society's endowment has been increased to include several identical copies of some objects. We also need to study allocations $\mathbf{X}$ where $\bigcup X_i$ is a strict subset of Ω. We view such cases as if society's endowment has been reduced to exclude some objects in Ω.

Let $\Omega(n)$ denote the set that contains n identical copies of each object $a \in \Omega$, and let $\mathbf{Z} := \{0, 1, ..., n\}^m$. We endow $\mathbf{Z}$ with the standard partial ordering: for $w, z \in \mathbf{Z}$, $w \preccurlyeq z$ iff $w_a \leqslant z_a$ for all $a \in \Omega$. With this partial ordering, $\mathbf{Z}$ is a lattice. Each $z \in \mathbf{Z}$ represents a bundle in $\Omega(n)$ which, for each $a \in \Omega$, contains z_a copies of a. We next extend the notion of an indicator vector, defined earlier. For any bundle $A \in \Omega(n)$, let $e^A \in \mathbf{Z}$ denote the *indicator vector*, whose coordinate e^A_a is equal to the number of copies of a contained in A, for each $a \in \Omega$. As before, if A is a singleton $\{a\}$, we will write sometimes e^a instead of e^A. Note that for any $z \in Z$, $z \wedge e^\Omega$ is a vector whose a-th coordinate is equal to 1 if $z_a \geqslant 1$ and equal to 0 otherwise. We change the domain of the utility function u_i from Ω to $\mathbf{Z}$ as follows: $u^*_i(e^A) := u_i(A)$ for each $A \subset \Omega$, and $u^*_i(z) := u^*_i(z \wedge e^\Omega)$ for any $z \in \mathbf{Z}$. With the change of domain, we can also extend the domain of the function u_i to $\Omega(n)$: for any $A \in \Omega(n)$, $u_i(A) := u^*_i(e^A)$. The interpretation of the extension u^*_i to vectors z having coordinates greater or equal to 2 is that buyer i's utility does not increase with the consumption of additional copies of the same object, no matter what other objects she is already consuming. It is easy to verify that if $u_i\colon \Omega \to \mathbf{R}$ satisfies the GS condition, then its extension $u_i\colon \Omega(n) \to \mathbf{R}$ satisfies the GS condition as well. Similarly, if u_i is monotone, its extension is monotone.

$\Omega(n)$ endowed with the set inclusion ordering ($A \preccurlyeq B$ iff $A \subset B$) is also a lattice. It is interesting to compare the lattices $\mathbf{Z}$ and $\Omega(n)$. In general, there are bundles $A \neq B$ for which $e^A = e^B$. Thus, $A \subset B$ implies $e^A \leqslant e^B$, but $e^A \leqslant e^B$ does not imply that $A \subset B$.

We now consider exchange economies $E^* = (z; u_1^*, ..., u_n^*)$ with total endowment $z \in \mathbf{Z}$. Alternatively, we consider economies $E' = (A; u_1, ..., u_n)$, with total endowment $A \subset \Omega(n)$. By the previous comment, if $u_i: \Omega \to \mathbf{R}$ has no complementarities for each $i \in N$, then all the results of Section 3 (especially, Lemma 6 and Theorem 3) apply to the economy $E' = (A; u_1, ..., u_n)$, for any $A \subset \Omega(n)$.

DEFINITION. The *surplus function* $S: \mathbf{Z} \to \mathbf{R}$ assigns to each m-dimensional *resource vector* z (with nonnegative integer coordinates) the value

$$S(z) := \max \left\{ \sum_{i \in N} u_i^*(z_i) \mid (z_1, ..., z_n) \in \mathbf{Z}^n \text{ and } \sum_{i \in N} z_i \leqslant z \right\}.$$

$S(z)$ is the total society value that can be achieved with a resource vector $z \in \mathbf{Z}$.

Prices for $\Omega(n)$ are of dimension nm, while prices for $\mathbf{Z}$ are of dimension m. Thus, when working with the domain $\mathbf{Z}$, we are implicitly assuming that all copies of an object have the same price. However, since different copies of an object are indistinguishable for the consumers, it is intuitive that even when prices for different copies are allowed to differ, in equilibrium these must coincide. But, this result is not used until Section 5, where we formally state it and prove it in Lemma 8.

THEOREM 4. *Suppose u_i is monotone and has the SI property for each $i \in N$. Let p be the smallest prices that support $A \subset \Omega$. Then, for each $a \notin A$, $p_a = S(e^A + e^a) - S(e^A)$. In particular, the smallest Walrasian prices are $\underline{p}_a = S(e^\Omega + e^a) - S(e^\Omega)$, $a \in \Omega$.*

Proof. Pick any $a \notin A$ and let $q_a := S(e^A + e^a) - S(e^A)$. Consider the economy $E' := (A \cup \{a\}; u_1, ..., u_n, u_a)$, where u_a denotes the unit demand preference defined by

$$u_a(b) = \begin{cases} \mu & \text{if } \quad b = a \\ 0 & \text{otherwise,} \end{cases}$$

and μ is a parameter to be specified.

Consider first the choice $\mu = q_a - \varepsilon$ for some $\varepsilon > 0$. Let $\mathbf{X}$ be an efficient allocation for E'. Since $\mu < q_a$, we must have that $X_a = \varnothing$. Thus p (restricted to $A \cup \{a\}$) together with $\mathbf{X}$ is a Walrasian equilibrium for E'. Therefore $p_a \geqslant \mu = q_a - \varepsilon$, and since this holds for any $\varepsilon > 0$, we conclude that $p_a \geqslant q_a$.

Now consider the choice $\mu = q_a + \varepsilon$ for some $\varepsilon > 0$. Again, let $(r, \mathbf{X})$ be a Walrasian equilibrium of E'. By efficiency, $X_a = \{a\}$, and therefore $r_a \leqslant \mu$. Let $M > u_i(\Omega)$ for all $i \in N$, and define $r_b := M$ for each $b \notin A \cup \{a\}$, to construct a price vector for Ω (which, with abuse of notation, we denote by the same symbol r). Then r supports A. Therefore $p_a \leqslant r_a \leqslant q_a + \varepsilon$ for all $\varepsilon > 0$. Hence $p_a \leqslant q_a$. ∎

THEOREM 5. *Suppose u_i is monotone and has the (SI) property for each $i \in N$. Then, $\bar{p}_a = S(e^{\Omega}) - S(e^{\Omega} - e^a)$ for every $a \in \Omega$.*

Proof. Pick any $a \in \Omega$ and define $q_a := S(e^{\Omega}) - S(e^{\Omega} - e^a)$. For u_a and μ as defined in the proof of Theorem 4, let $E' := (\Omega; u_1, ..., u_n, u_a)$. We first show that $\bar{p}_a \leqslant q_a$ by contradiction. Suppose that $\bar{p}_a > q_a$ and let $\mu := (q_a + \bar{p}_a)/2$. Let $(\bar{p}, \mathbf{X})$ be a Walrasian equilibrium of E, and $\mathbf{X}' := (\mathbf{X}, X_a)$, where $X_a := \varnothing$. Since $\bar{p}_a > \mu$, $(p, \mathbf{X}')$ is a Walrasian equilibrium of E'. By the first theorem of welfare economics, the maximal social surplus in E' is equal to that of E (that is, equal to $S(e^{\Omega})$). But if instead we allocate optimally $\Omega \backslash \{a\}$ among the first n consumers and give a to the last consumer, then the total surplus is $S(e^{\Omega} - e^a) + \mu > S(e^{\Omega} - e^a) + q_a = S(e^{\Omega})$, which is a contradiction.

Now make $\mu := q_a$. Let $\mathbf{X}$ be an efficient allocation in E' such that $X_a = \varnothing$. By Lemma 6, $\bar{p}$ supports $\mathbf{X}$. Therefore $\bar{p}_a \geqslant \mu = q_a$. ∎

Theorems 4 and 5 generalize Leonard's [8] results for unit demand economies.

THEOREM 6. *Suppose each u_i is monotone and has the SI property. Then, $S: \mathbf{Z} \to \mathbf{R}$ has decreasing marginal returns.*

Proof. It is easy to show that S is submodular iff for any $z \in \mathbf{Z}$ such that for two elements $a, b \in \Omega$, $z_a, z_b < n$,

$$S(z + e^a) + S(z + e^b) \geqslant S(z) + S(z + e^a + e^b).$$

Suppose $z \in \mathbf{Z}$ and $a, b \in \Omega$ satisfy the above conditions. Let $A, B \subset \Omega(n)$ be such that $e^A = z$ and $e^B = z + e^a + e^b$. Define $\mu_a := S(z + e^x) - S(z)$ and $\mu_b := S(z + e^b) - S(z)$. Consider the economy $E' := (B; u_1, ..., u_n, u_a, u_b)$, where the last two consumer have unit demand preferences and only care about objects a and b respectively. Allocate A efficiently among consumers $i \in N$, and give a to consumer a and b to consumer b; call this allocation $\mathbf{X}$. If p denotes the smallest prices that supports A (in the economy $E^n := (\Omega(n); u_1, ..., u_n)$), then $(p, \mathbf{X})$ is a Walrasian equilibrium of E'. By the definition of μ_a, another efficient allocation in E' can be constructed by allocating $A \cup \{a\}$[1] efficiently among consumers $i \in N$, give nothing to con-

[1] Here $A \cup \{a\}$ denotes the bundle in $\Omega(n)$ that contains one more copy of a than A, and the same number of copies of x for any other $x \in \Omega$.

sumer a and b to consumer b. By Lemma 6, p together with this new allocation is also a Walrasian equilibrium of E'. This implies that p also supports $A \cup \{a\}$ (in E'') and that $p_b \leqslant \mu_b$. By Theorem 4, $p_b \geqslant S(e^A + e^a + e^b) - S(e^A + e^a)$. Thus $S(e^A + e^b) - S(e^A) \geqslant S(e^A + e^a + e^b) - S(e^A + e^a)$, as was to be shown. ∎

Suppose $u_i\colon 2^\Omega \to \mathbf{R}$ is monotone and has the SI property for each $i \in N$. Lemma 1 then implies that $S\colon \mathbf{Z} \to \mathbf{R}$ is submodular. It can be shown, however, that if each u_i is only submodular (and monotone), S may fail to be submodular.

The following comparative static result is reminiscent of Topkis' monotonicity theorem [11]. View the Walrasian equilibrium problem as parametrized by the set of objects available in the economy, and endow 2^Ω (the set of parameters) with the partial order $A \preccurlyeq B$ iff $A \subset B$. As in Topkis' theorem, for each parameter A, the set of solutions P^A (Walrasian prices) associated with A is a complete lattice, and if $A \preccurlyeq B$, then $\bigwedge P^A \geqslant \bigwedge P^B$ and $\bigvee P^A \geqslant \bigvee P^B$. A similar result holds if we view the Walrasian equilibrium problem as parametrized by the set of consumers.

THEOREM 7. *Suppose each u_i is monotone and has the SI property. For each bundle $A \subset \Omega$ and consumer i define the economies $E^A := (A; u_1, ..., u_n)$ and $E^{-i} := (\Omega; u_{-i})$. Let $\underline{p}^A$ and $\bar{p}^A$ denote respectively the smallest and largest equilibrium price vector for E^A. Define $\underline{p}^{-i}$ and $\bar{p}^{-i}$ similarly for E^{-i}. Then* (i) *if $A \subset B \subset \Omega$, $\underline{p}^A_a \geqslant \underline{p}^B_a$ and $\bar{p}^A_a \geqslant \bar{p}^B_a$ for all $a \in A$; and* (ii) *$\underline{p}^{-i} \leqslant \underline{p}$ and $\bar{p}^{-i} \leqslant \bar{p}$.*

Proof. (i) follows directly from Theorems 4, 5, and 6.

Next we show that $\underline{p}^{-i} \leqslant \bar{p}$. Denote by S_i the surplus function associated with E^{-i}. Let $(\underline{p}^{-i}, \mathbf{X}_{-i})$ be a Walrasian equilibrium for E^{-i}, and suppose the bundle X_i maximizes $v_i(B, \underline{p}^{-i})$ over all $B \subset \Omega$. Then, $(\underline{p}^{-i}, \mathbf{X})$ is a Walrasian equilibrium for the economy $E' = (A; u_1, ..., u_n)$, where $A \subset \Omega(2)$ is the bundle that contains two copies of each object in X_i and one copy of all other objects in $\Omega \backslash X_i$. By part (i), for each $a \in \Omega$,

$$\begin{aligned}\underline{p}^{-i}_a &= S_i(e^\Omega + e^a) - S_i(e^\Omega) = S_i(e^\Omega + e^a) + u_i(X_i) - [S_i(e^\Omega) + u_i(X_i)]\\ &= S_i(e^\Omega + e^a) + u_i(X_i) - S(e^\Omega + e^{X_i})\\ &\leqslant S(e^\Omega + e^{X_i} + e^a) - S(e^\Omega + e^{X_i}) \leqslant \underline{p}_a.\end{aligned}$$

The second inequality in (ii) is proved analogously. ∎

With the two additional conditions NTW and NTF discussed earlier, Kelso and Crawford's [5] Theorem 5 establishes for their discretized economy results similar to our Theorem 7.

5. VICKREY AUCTIONS

In this section we compare the outcomes of strategy-proof mechanisms studied by Vickrey [12], Clare [3] and Groves [4] with Walrasian outcomes. In particular, we show that the Vickrey–Clarke–Groves (VCG) payment for a bundle is never greater than the value of that bundle at the smallest Walrasian prices. Moreover, we show that the gap between these two values disappears if the economy is replicated at least $m+1$ times, where m is the number of different objects. Throughout this section we assume that agent 0 (the seller) initially owns all the objects and has no utility for them.

In discussing mechanism design issues, we need to consider the possibility that agents do not report truthfully, and thus, temporarily, we make explicit the dependence of the surplus function on the profile of utilities.

Definition. For a given profile of preferences $u=(u_1, ..., u_n)$ over Ω, $z \in \mathbf{Z}$, and consumer i, let

$$S(z; u) := \max \left\{ \sum_{j \in N} u_j(X_j) \,\middle|\, \sum_{j \in N} e^{X_j} \leqslant z \right\}$$

$$S_i(z; u_{-i}) := \max \left\{ \sum_{j \neq i} u_j(X_j) \,\middle|\, \sum_{j \in N} e^{X_j} \leqslant z \right\}.$$

Vickrey auction. Each buyer i submits a *complete* utility function $u_i \colon 2^{\Omega} \to \mathbf{R}$ (this is equivalent to reporting a vector of dimension $2^m - 1$). The seller then finds an efficient allocation $\mathbf{X}$ with respect to the reported profile of preferences $(u_1, ..., u_n)$. Consumer $i \in N$ receives the bundle X_i and pays the Vickrey payment $q_i(X_i; u_{-i})$, where

$$q_i(X_i; u_{-i}) := S_i(e^{\Omega}; u_{-i}) - S_i(e^{\Omega} - e^{X_i}; u_{-i}).$$

Note that the Vickrey payments depend on the efficient allocation chosen, and that there might be several efficient allocations associated with the same utility profile $u=(u_1, ..., u_n)$. It is well known that the buyers and the seller are indifferent about which efficient allocation is chosen when every buyer reports his true preferences (see, for example, Krishna and Perry [7]), and that the VCG mechanism is strategy-proof. For the rest of this section, we assume that the agents report truthfully their preferences, and drop the profile u form the list of arguments in the functions S, S_i, and q_i, $i \in N$.

Consider the case where all consumers have unit demand preferences. We only need to consider allocations that assign at most one object to each consumer. Thus, the Vickrey payments are defined for *objects*. Leonard [8] has shown that in this case, the Vickrey payments coincide with the smallest Walrasian prices. It is easy to extend this result to the case in which each consumer i has "linear preferences" of the form

$$u_i(A) = \sum_{a \in A} u_i(\{a\}), \qquad A \subset \Omega.$$

However, the following example shows that with more general utility functions that have the SI property, this result typically does not hold. There are three identical objects and two consumers with the same preferences. For $i = 1, 2$, $u_i(A)$ is equal to 0 if $\#(A) = 0$, to 10 if $\#(A) = 1$, to 18 if $\#(A) = 2$, and to 20 if $\#(A) = 3$. Since the objects are indistinguishable, in any equilibrium their prices must coincide. All efficient allocations assign one object to one consumer and two objects to the other. Therefore, (8, 8, 8) is the unique Walrasian price vector. The Vickrey payment for the consumer getting one object is $20 - 18 = 2 < 8$, and for the consumer getting two objects is $20 - 10 = 10 < 8 + 8 = 16$. Thus, each consumer is paying *strictly less* in the Vickrey auction than in (any) Walrasian equilibrium. Although the equality is not attained in general, the next theorem establishes that even without the GS condition, a consumer's Vickrey payment for her bundle is never more than the value of that bundle at the smallest Walrasian prices.

THEOREM 8. *Let $(p; \mathbf{X})$ be a Walrasian equilibrium of $E = (\Omega; u_1, ..., u_n)$. Suppose each u_i is monotone. Then, $\langle p, X_i \rangle \geqslant q_i(X_i)$ for each $i \in N$.*

Proof. Consider the economy $E' = (\Omega; u_1, ..., u_{i-1}, u'_i, u_{i+1}, ..., u_n)$, where consumer i is replaced by a consumer with linear preferences, given by

$$u'_i(A) = \sum_{a \in A \cap X_i} p_a.$$

It is easy to see that $(p, \mathbf{X})$ is also a Walrasian equilibrium of E', with associated total surplus

$$S' = \langle p, X_i \rangle + S_i(e^{\Omega} - e^{X_i}).$$

Now consider the economy E'', where consumer i is replaced by a consumer with utility function $u''_i(A) = 0$ for all A. Its total surplus is $S'' = S_i(e^{\Omega})$. Obviously $S' \geqslant S''$, and by definition, $S_i(e^{\Omega}) = q_i(X_i) + S_i(e^{\Omega} - e^{X_i})$. Hence, $\langle p, X_i \rangle \geqslant q_i(X_i)$. ∎

Makowski and Ostroy [9] prove that in a quasilinear economy the private marginal product of each agent is no greater than his social marginal product. Straightforward manipulations of their definitions of private and social marginal product reveal this result to be equivalent to Theorem 8 above.

The next results deal with replica economies. For any $k \in \mathbf{N}$, the *k-replica* of economy $E = (\Omega; u_1, ..., u_n)$ is the economy $\hat{E}$ with set of objects $\Omega(k)$ containing k identical copies of each object in Ω, and k "copies" $i1, ..., ik$ of each consumer i. The utility function of a consumer ij is defined as follows. For any bundle $A \subset \Omega(k)$, let $u_{ij}(A) := u_i^*(e^A \wedge e^\Omega)$ (as defined in the previous section).[2]

If $\mathbf{X}$ is an allocation for E, then $\mathbf{X}^k$ denotes the allocation for $\hat{E}$ in which each consumer type i receives the same bundle X_i. If $\hat{p}$ is a price vector in $\hat{E}$, then $\hat{p} = (\hat{p}^1, ..., \hat{p}^k)$, where $\hat{p}^j$ represents the prices for the j-the copy of each object.

LEMMA 7. *Suppose u_i is monotone and has no complementarities for each $i \in N$. If $\mathbf{X}$ is an efficient allocation for E, then $\mathbf{X}^k$ is an efficient allocation for its k-replica economy $\hat{E}$, and if p is a Walrasian price vector for E, then $\hat{p} = (p, ..., p)$ is a Walrasian price vector for $\hat{E}$. Conversely, if $\hat{p}$ is a Walrasian price vector for $\hat{E}$, there exists a price vector p in E such that $\hat{p} = (p, ..., p)$.*

Proof. Let $\mathbf{X}$ be an efficient allocation for E. By Lemma 6, if p is any Walrasian price vector for E, then $(p, \mathbf{X})$ is a Walrasian equilibrium. It is easy to see that $((p, ..., p), \mathbf{X}^k)$ is a Walrasian equilibrium for $\hat{E}$. Therefore, by the first theorem of welfare economics, $\mathbf{X}^k$ is an efficient allocation for $\hat{E}$.

Now, suppose $(\hat{p}, \hat{\mathbf{X}})$ is a Walrasian equilibrium for $\hat{E}$. Then, by the definition of the preferences of the consumers in $\hat{E}$, we can assume wlog that for each ij, $\hat{X}_{ij}$ contains at most one copy of a, for every $a \in \Omega$. Therefore, for any two copies a' and a'' of the same object a, $\hat{p}_{a'} = \hat{p}_{a''}$, for otherwise the individual consuming the most expensive copy would rather consume the cheapest copy instead. ∎

Since we will only consider Walrasian prices $\hat{p}$ for $\hat{E}$, by Lemma 7 and with abuse of notation we will view $\hat{p}$ as an m-dimensional vector only.

COROLLARY 2. $\hat{P}^W = P^W$.

[2] Although in Section 4 we only required the case $k = n$, it is clear that the preferences' extensions and the notation defined there apply to any $k \geq 2$.

We have proven above that the smallest Walrasian prices $\underline{p}$ are always an upper bound for the set of Vickrey payments, in the sense that the value of consumer i's bundle at prices $\underline{p}$ is never less then its corresponding Vickrey payment. The next theorem shows that if the economy is replicated at least $k = m+1$ times, then the Walrasian prices "coincide" with the Vickrey payments. For each consumer ij of $\hat{E}$, we denote by $\hat{S}$ and $\hat{S}_{ij}$ the surplus functions of $\hat{E}$, and for any bundle $A \subset \Omega$, let $\hat{q}_{ij}(A) := \hat{S}_{ij}(ke^{\Omega}) - \hat{S}_{ij}(ke^{\Omega} - e^{A})$ (note that $e^{\Omega(k)} = ke^{\Omega}$). As before, we omit the profile of preferences because it is assumed to be fixed at the true profile.

THEOREM 9. *Suppose that u_i is monotone and has no complementarities for each $i \in N$. Let $k \geqslant m+1$ and $\hat{E}$ be the k-replica economy. If $\hat{\mathbf{X}}$ is any efficient allocation of $\hat{E}$ (not necessarily of the form $\mathbf{X}^k$ for some efficient allocation $\mathbf{X}$ of E), then $\hat{q}_{ij}(\hat{X}_{ij}) = \langle \underline{p}, \hat{X}_{ij} \rangle$ for each replica ij of consumer i, and each $i \in N$.*

Proof. Let $\mathbf{Y}$ be any efficient allocation in E, and pick any consumer in $\hat{E}$; without loss of generality, and to simplify the notation, assume this is a consumer rk in the last cohort. By Lemma 7, we have

$$k \cdot S(e^{\Omega}) = \hat{S}(e^{\Omega(k)}) = \hat{S}_{rk}(e^{\Omega(k)} - e^{\hat{X}_{rk}}) + u_{rk}(\hat{X}_{rk}). \tag{1}$$

One possible allocation of $\Omega(k)$ among the consumers excluding rk is obtained as follows. Suppose $Y_r = \{a_1, ..., a_l\}$. For each $j = 1, ..., l$, distribute $\Omega \cup \{a_j\}$ efficiently among the consumers in the jth cohort. For each $j = l+1, ..., k-1$, give each consumer ij, $i \in N$, the bundle Y_i. Finally, for the last cohort, give each consumer ik, excluding consumer rk, the bundle Y_i. This allocation has total surplus

$$\begin{aligned}&\sum_{j=1}^{l} S(e^{\Omega} + e^{a_j}) + (k-l-1)\, S(e^{\Omega}) + [S(e^{\Omega}) - u_r(Y_r)]\\ &= kS(e^{\Omega}) + \langle \underline{p}, Y_r \rangle - u_r(Y_r).\end{aligned}$$

Therefore,

$$\hat{S}_{rk}(\Omega(k)) \geqslant kS(e^{\Omega}) + \langle \underline{p}, Y_r \rangle - u_r(Y_r), \tag{2}$$

and

$$\begin{aligned}\hat{q}_{rk}(\hat{X}_{rk}) &= \hat{S}_{rk}(e^{\Omega(k)}) - \hat{S}_{rk}(e^{\Omega(k)} - e^{\hat{X}_{rk}})\\ &\geqslant [kS(e^{\Omega}) + \langle \underline{p}, Y_r \rangle - u_r(Y_r)] - [kS(e^{\Omega}) - u_{rk}(\hat{X}_{rk})]\\ &= u_{rk}(\hat{X}_{rk}) - u_r(Y_r) + \langle \underline{p}, Y_r \rangle,\end{aligned}$$

where the last inequality follows from (1) and (2). Since $\hat{\mathbf{X}}$ is efficient, wlog we can assume that $\hat{X}_{rk} \subset \Omega$, and therefore $u_{rk}(\hat{X}_{rk}) = u_r(\hat{X}_{rk})$. Again by Lemma 7, $(\underline{p}, \mathbf{Y})$ and $(\underline{p}, \hat{\mathbf{X}})$ are respectively Walrasian equilibria of E and $\hat{E}$. Thus

$$u_r(Y_r) - \langle \underline{p}, Y_r \rangle = u_r(\hat{X}_{rk}) - \langle \underline{p}, \hat{X}_{rk} \rangle.$$

Substituting this equality in the previous inequality, we get $\hat{q}_{rk}(\hat{X}_{rk}) \geqslant \langle \underline{p}, \hat{X}_{rk} \rangle$. Theorem 8 then implies that $\hat{q}_{rk}(\hat{X}_{rk}) = \langle \underline{p}, \hat{X}_{rk} \rangle$. ∎

6. PRODUCTION

In this section we introduce a production technology that satisfies a condition, no complementarities in production (NCP), analogous to the NC condition. We show how a production economy endowed with this technology can be identified with an exchange economy satisfying the GS condition. We then use this construction to extend results from preceding sections to economies with production. Suppose that there are ℓ firms in the economy, and define $L := \{1, ..., \ell\}$. Let Ω denote the maximal collection of objects (including multiple units or *copies*) that the agents would ever consume collectively. Without any assumptions, the set Ω may be infinite. We will assume below that production costs are "convex", and that there exists a set Ω sufficiently large so that the marginal surplus for the consumers (when they consume Ω) of any additional (unit of an) object is less than the marginal cost of producing that object (when the firms are already producing Ω efficiently). As before, $m := \#(\Omega)$.

Firm k is totally characterized by its *cost function* $c_k: 2^\Omega \to \mathbf{R}_+$. We will require that each c_k be monotone and have no production complementarities (as defined below).

DEFINITION. Firm k's *profit function* $\pi_k: 2^\Omega \times \mathbf{R}^m_+ \to \mathbf{R}$ and *supply correspondence* $\Sigma_k: \mathbf{R}^m_+ \to 2^\Omega$ are defined by

$$\pi_k(A, p) := \langle p, A \rangle - c_k(A) \qquad A \subset \Omega, \quad p \in \mathbf{R}^m_+,$$

$$\Sigma_k(p) := \{A \mid \pi_k(A, p) \geqslant \pi_k(B, p) \text{ for all } B \subset \Omega\} \qquad p \in \mathbf{R}^m_+.$$

DEFINITION. The cost function $c_k: 2^\Omega \to \mathbf{R}_+$ is *monotone* if $c_k(A) \geqslant c_k(B)$ for all $A \supset B$, and has *no production complementarities* (NPC) if for every $A, B \in \Sigma_k(p)$ and $X \subset A \backslash B$, there exists $Y \subset B \backslash A$ such that $[A, X, Y] \in \Sigma_k(p)$.

DEFINITION. $(p; X_1, ..., X_n; Y_1, ..., Y_\ell)$ is a Walrasian equilibrium for the production economy $E^P = (\Omega; u_1, ..., u_n; c_1, ..., c_\ell)$ if

(1) $v_i(X_i, p) \geqslant v_i(A, p)$ for all $A \subset \Omega$ and $i \in N$.

(2) $\pi_k(Y_k, p) \geqslant \pi_k(A, p)$ for all $A \subset \Omega$ and $k \in L$.

(3) $\sum_{i \in N} e^{X_i} \leqslant \sum_{k \in L} e^{Y_k}$ (the difference of these two vectors represents the set of objects that are produced and not consumed).

A firm that produces $A^c = \Omega \backslash A$ can be viewed as "consuming" the bundle A. To construct a Walrasian equilibrium, we will transform every firm into a consumer and expand the set of objects in the economy to $\Omega(\ell)$.

DEFINITION. Firm k's *utility function* u_k^P, *net utility function* v_k^P, and *demand correspondence* D_k^P are defined as follows:

$$\begin{aligned} u_k^P(A) &:= c_k(\Omega) - c_k(A^c) \qquad A \subset \Omega, \\ v_k^P(A, p) &:= u_k^P(A) - \langle p, A \rangle \\ &= c_k(\Omega) - c_k(A^c) - \langle p, A \rangle \qquad A \subset \Omega, \quad p \in \mathbf{R}_+^m, \\ D_k^P(p) &:= \{ A \mid v_k^P(A, p) \geqslant v_k^P(B, p) \text{ for all } B \subset \Omega \} \qquad p \in \mathbf{R}_+^m. \end{aligned}$$

Clearly, $\pi_k(A^c, p) - \pi_k(\Omega, p) = v_k^P(A, p)$. Thus, A maximizes firm k's net utility at prices p iff A^c maximizes firm k's profits at prices p. That is, $A \in D_k^P(p)$ iff $A^c \in \Sigma_k(p)$.

LEMMA 8. *If c_k has no production complementarities then u_k^P has no complementarities, and if c_k is monotone then u_k^P is monotone.*

Proof. Suppose c_k is monotone and $A \supset B$. Then $A^c \subset B^c$ and $u_k^P(A) = c_k(\Omega) - c_k(A^c) \geqslant c_k(\Omega) - c_k(B^c) = u_k^P(B)$. Hence u_k^P is monotone.

Now, suppose $A, B \in D_k^P(p)$ and $X \subset A \backslash B$. Then $A^c, B^c \in \Sigma_k(p)$ and $X \subset B^c \backslash A^c$. Let $\hat{X} := A \cap B^c \cap X^c$; clearly $\hat{X} \subset B^c \backslash A^c$. By (NPC), there exists $\hat{Y} \subset A^c \backslash B^c = B \backslash A$ such that $[B^c, \hat{X}, \hat{Y}] \in \Sigma_k(p)$. But,

$$\begin{aligned} [B^c, \hat{X}, \hat{Y}]^c &= [(B^c \cap \hat{X}^c) \cup \hat{Y}]^c = (B \cup \hat{X}) \cap \hat{Y}^c \\ &= [B \cup (A \cap B^c \cap X^c)] \cap \hat{Y}^c \\ &= [(A \cup B) \backslash X] \cap \hat{Y}^c = [A, X, (B \backslash \hat{Y})] \end{aligned}$$

Let $Y := B \backslash \hat{Y} \subset B$; then $[A, X, Y] = [B^c, \hat{X}, \hat{Y}]^c \in D_k^P(p)$. ∎

View $\Omega(\ell)$ as $\bigcup \Omega_k^*$, where each Ω_k^* is a *different* copy of Ω. Ω_k^* represents the production set of firm k. Let Π: $\Omega(\ell) \to \Omega$ be the *projection* map that to any k and "copy" $a_k^* \in \Omega_k^*$ of $a \in \Omega$, assigns its "original" object a. Then, for any $A \subset \Omega(\ell)$ and k, let $A_k^* := A \cap \Omega_k^*$ and $A_k := \Pi(A_k^*) \subset \Omega$.

We extend the consumers' utilities as in Section 4. For each $A \subset \Omega(\ell)$ and $i \in N$, $u_i(A) := u_i(\Pi(A))$. The producers' utilities on $\Omega(\ell)$, however, are defined in a different way: $u_k^P(A) := u_k^P(A_k)$ for any $A \subset \Omega(\ell)$ and $k \in L$. If the original u_k^P is monotone and/or has no complementarities, the new u_k^P just defined has the same properties. We are abusing notation here, since we denote by the same symbol the utilities on Ω and on $\Omega(\ell)$. Notice that the producers utilities on $\Omega(\ell)$ are not "extensions" of their utilities on Ω: producer k has positive utility only for copies in Ω_k^*.

To study the existence and properties of Walrasian equilibria of the production economy, we consider the *exchange* economy $E = (\Omega(\ell); u_1, ..., u_n, u_1^P, ..., u_\ell^P)$. In this exchange economy we refer to agent i ($i = 1, ..., n$) as consumer i, and to agent $n+k$ ($k = 1, ..., \ell$) as producer k.

THEOREM 10. *Assume each u_i and each c_k in the economy E^P is monotone and has no (production) complementarities. Then E^P has a Walrasian equilibrium. Moreover, the set of Walrasian equilibrium prices for E^P is a complete lattice.*

Proof. By Kelso and Crawford [5], the exchange economy E has a Walrasian equilibrium $((p^1, ..., p^\ell); X_0^*, ..., X_n^*, X_1^P, ..., X_\ell^P)$. Since each producer k assigns no value to objects outside Ω_k^*, we can assume without loss of generality that $X_k^P \subset \Omega_k^*$ for each k. Let $Y_k := \Pi(\Omega_k^* \backslash X_k^P)$, $k \in L$. Then, $e^{Y_k} = e^{\Omega} - e^{X_k^p}$ (recall that for any $B \subset \Omega(\ell)$ and $a \in \Omega$, e_a^B is the number of copies of a contained in B). Similarly, let $X_i := \Pi(X_i^*)$, $i \in N$. By the way the consumers' preferences are extended to $\Omega(\ell)$, we can assume that $e^{X_i} = e^{X_i^*}$ for each $i \in N$ (that is, each consumer does not get more than a copy of each object). Now, since $(X_0^*, ..., X_n^*, X_1^P, ..., X_\ell^P)$ is a partition of $\Omega(\ell)$,

$$\ell e^{\Omega} = \sum_{i \in N_0} e^{X_i^*} + \sum_{k \in L} e^{X_k^P} = e^{X_0^*} + \sum_{i \in N} e^{X_i} + \sum_{k \in L} [e^{\Omega} - e^{Y_k}].$$

That is,

$$\sum_{k \in L} e^{Y_k} = e^{X_0^*} + \sum_{i \in N} e^{X_i} \geqslant \sum_{i \in N} e^{X_i}.$$

The price vector p^k denotes the prices of objects in Ω_k^* (charged by producer k). Since consumers $i \in N$ find the objects in Ω_k^* equivalent to the objects in Ω_j^* for any $j \neq k$, we must have $p^k = p^j$. Suppose to the contrary that for some $j \neq k$ and object a, $p_a^k > p_a^j \geqslant 0$. Then object a_k^* is consumed by some consumer i (because its price is positive). But i is indifferent between a_j^* and a_k^*, and a_j^* is cheaper; this is a contradiction. Therefore $p^k = p^1$ for all $k > 1$.

Since X_i^* is optimal for i in $\Omega(\ell)$ at prices $(p^1, ..., p^1)$, X_i is optimal for i in Ω at prices p^1. Similarly, X_k^P is optimal for k in $\Omega(\ell)$ at prices $(p^1, ..., p^1)$ iff Y_k is profit maximizing in Ω for firm k at prices p^1. Therefore, $(p^1; X_1, ..., X_n; Y_1, ..., Y_\ell)$ is a Walrasian equilibrium of the production economy E^P.

It is straightforward to extend to production economies Theorem 3 and Corollary 1. ∎

7. CONCLUSION

In this paper we have studied the problem of efficient production and allocation when the commodity space consists of m indivisible goods and one divisible good (money). The key assumptions are the quasilinearity in the divisible good, the GS condition, and that each consumer is endowed with a sufficient amount money.

Within this setting, we were able to provide an analysis of Walrasian equilibrium. We also established a relationship between Walrasian equilibrium and strategy-proof mechanisms. Two of the three main assumptions of the model that have been developed in this paper are familiar from auction theory. Quasilinearity in money and the fact that agents are endowed with a significant amount of money are standard assumptions in the literature. In a companion paper we build on this connection to auction theory, and study a dynamic auction/tâtonnement process when preferences and cost functions satisfy the GS condition.

8. APPENDIX

The next three lemmas show that if u is monotone, then the three conditions GS, NC, and SI are equivalent.

Lemma 2. *If u is monotone, GS implies SI.*

Proof. Pick a price vector p, and let $A \notin D(p)$. For any price vector q define

$$H(q) := \{B \subset \Omega \mid v(B, q) > v(A, q)\}$$

$$H_1(q) := \{B \in H(q) \mid \#(B \backslash A) \leqslant \#(C \backslash A) \text{ for all } C \in H(q)\}$$

$$H_2(q) := \{B \in H(q) \mid \#(A \backslash B) \leqslant \#(A \backslash C) \text{ for all } C \in H(q)\}.$$

$H(q)$ are the bundles that have strictly higher net utility then A at prices q. Since $A \notin D(p)$, $D(p) \subset H(p)$, and therefore $H_1(p)$ is nonempty. We first show that for any $B \in H_1(p)$, $\#(B \backslash A) \leqslant 1$.

Let $B \in H_1(p)$ and $\hat{p}$ be the price vector such that $\hat{p}_a = p_a$ for all $a \in A \cup B$ and $\hat{p}_a = M > u(\Omega)$ otherwise. Observe that $\emptyset \neq D(\hat{p}) \subset H(\hat{p}) \subset H(p)$. Pick any $C \in H(\hat{p})$. Then $C \subset A \cup B$, and therefore $C \backslash A \subset B \backslash A$. That is, $C \in H_1(p)$. Hence $H(\hat{p}) \equiv H_1(\hat{p}) \subset H_1(p)$. To conclude, it is enough to show that $\#(C \backslash A) \leqslant 1$. By contradiction, suppose $\#(C \backslash A) \geqslant 2$. Pick $\{x, y\} \subset C \backslash A$, and for each $\varepsilon \geqslant 0$, let $q(\varepsilon) := \hat{p} + \varepsilon e^{\{x, y\}}$. Let

$$\Delta := \{\varepsilon \geqslant 0 \mid A \notin D(q(\varepsilon)) \text{ and } C \in D(q(\varepsilon))\}.$$

Since $q(0) = \hat{p}$, $0 \in \Delta$. Let $\bar{\varepsilon} := \sup \Delta$. Since D is upper semicontinuous, $C \in D(q(\bar{\varepsilon}))$.

There are two possibilities at $\bar{\varepsilon} =$ either $A \notin D(q(\bar{\varepsilon}))$ or $A \in D(q(\bar{\varepsilon}))$. Assume first the former. Then, there exists $\hat{\varepsilon} > \bar{\varepsilon}$ such that $A, C \notin D(q(\hat{\varepsilon}))$. Pick any $X \in D(q(\hat{\varepsilon}))$; then $X \subset A \cup B$, and either $x \notin X$ or $y \notin X$ (or both). Moreover, $v(X, p) \geqslant v(X, q(\hat{\varepsilon})) > v(A, q(\hat{\varepsilon})) = v(A, p)$. Therefore, $X \in H(p)$ and $\#(X \backslash A) \leqslant \#(C \backslash A) - 1$, which contradicts the fact that $C \in H_1(p)$. Alternatively, now assume that $A \in D(q(\bar{\varepsilon}))$. Let $r(\varepsilon) := q(\bar{\varepsilon}) + \varepsilon e^x$. Note that for all $\varepsilon > 0$, $A \in D(r(\varepsilon))$ and $C \notin D(r(\varepsilon))$. By GS, there exists X such that $C \backslash \{x\} \subset X \in D(r(\varepsilon))$ for all $\varepsilon \geqslant 0$. Moreover, since $C \notin D(r(\varepsilon))$ for all $\varepsilon > 0$, $x \notin X$. Pick any $\varepsilon > 0$; then $v(A, p) = v(A, r(\varepsilon)) = v(X, r(\varepsilon))$, and since $y \in X$, $v(X, r(\varepsilon)) < v(X, p)$. Consequently, $X \in H(p)$ and $\#(X \backslash A) \leqslant \#(C \backslash A) - 1$, which is again a contradiction. We have thus shown that $\#(B \backslash A) \leqslant 1$ for all $B \in H_1(p)$.

For the rest of the proof, fix $B \in H_1(p)$ and define $\hat{p}$ as follows: $\hat{p}_a = p_a$ for all $a \in A \cup B$ and $\hat{p}_a = M > u(\Omega)$ otherwise.

Pick $E \in H_2(\hat{p})$ and define p^0 as follows: $p^0_a = 0$ for $a \in A \cap E$, and $p^0_a = \hat{p}_a$ otherwise. Recall that $H(\hat{p}) \equiv H_1(\hat{p}) \subset H_1(p)$. Therefore, $H_2(\hat{p}) \subset H_1(\hat{p})$, and thus $E \in H_1(\hat{p})$ as well. Hence, $E \backslash A = B \backslash A$. To finish the proof, we show that $\#(A \backslash E) \leqslant 1$. More specifically, assume that $\#(A \backslash E) > 1$; we then show that there exists $G \in H_1(\hat{p})$ such that $\#(A \backslash G) < \#(A \backslash E)$, which is a contradiction.

Observe that if $X \in H(p^0)$, then $0 < v(X, p^0) - v(A, p^0) \leqslant v(X, \hat{p}) - v(A, \hat{p})$. So $X \in H(\hat{p}) \equiv H_1(\hat{p})$, and therefore $H(p^0) \equiv H_1(p^0) \subset H_1(\hat{p})$. Thus, $X \backslash A = E \backslash A = B \backslash A$. Also, $E \in H_1(p^0)$ and $A \notin D(p^0)$.

We now show that $E \in D(p^0)$. By contradiction, suppose that $v(X, p^0) > v(E, p^0)$; let $X^* := X \cup (A \cap E)$. Then, $v(X^*, p^0) > v(E, p^0)$. Hence, $X^* \neq E$, $X^* \subset A \cup E$, and $A \cap E \subset X^*$. Thus, $\#(A \backslash X^*) < \#(A \backslash E)$, contradicting the fact that $E \in H_2(p^0)$. Therefore, $E \in D(p^0)$.

For $\varepsilon \geqslant 0$, let $q(\varepsilon) = p^0 - \varepsilon e^{A \backslash E}$. Define

$$\Delta := \{\varepsilon \geqslant 0 \mid q(\varepsilon) \geqslant 0,\ A \notin D(q(\varepsilon)), \text{ and } E \in D(q(\varepsilon))\}.$$

Since $E \in H_2(p^0)$, $p_a^0 > 0$ for all $a \in A \backslash E$. Also, since $A \notin D(p^0)$, $A \notin D(q(\varepsilon))$ for all $\varepsilon > 0$ sufficiently small. Let $\bar{\varepsilon} := \sup \Delta$. We show that $\bar{\varepsilon} > 0$. If not, $E \notin D(q(\varepsilon))$ for all $\varepsilon > 0$, and therefore there exists F such that $v(F, q(\varepsilon)) > v(E, q(\varepsilon))$ for all $\varepsilon > 0$ sufficiently small. Let $F^* := F \cup (A \cap E)$. Then, $v(F^*, q(\varepsilon)) > v(E, q(\varepsilon))$ for all small $\varepsilon > 0$. By continuity, since $E \in D(p^0)$ and $q(0) = p^0$, $F^* \in D(p^0) \subset H(p^0)$. Also $F^* \neq E$, and by earlier remark, $F^* \backslash A = E \backslash A$. Therefore, $\#(A \backslash F^*) < \#(A \backslash E)$, which contradicts the fact that $E \in H_2(p^0)$. Hence, $\bar{\varepsilon} > 0$.

At $\bar{\varepsilon}$ one of three things happen: (i) $q_x(\bar{\varepsilon}) = 0$ for some $x \in A \backslash E$; (ii) $A \notin D(q(\bar{\varepsilon}))$ and $E \notin D(q(\varepsilon))$ for all $\varepsilon > \bar{\varepsilon}$; (iii) $A \in D(q(\bar{\varepsilon}))$. In case (i), make $G := E \cup \{x\}$. In case (ii), there exists $G \in D(q(\bar{\varepsilon}))$ such that $A \cap E \subset G \subset A \cup B$ and $G \neq E$. In these two cases $v(G, q(\bar{\varepsilon})) = v(E, q(\bar{\varepsilon})) > v(A, q(\bar{\varepsilon}))$ and $\#(A \backslash G) < (A \backslash E)$. But, $v(G, q(\bar{\varepsilon})) - v(A, q(\bar{\varepsilon})) = v(G, p^0) - v(A, p^0)$, so $G \in H(p^0)$, which contradicts the fact that $E \in H_2(p^0)$.

Finally, in case (iii), $A \in D(q(\bar{\varepsilon}))$. Since by assumption $\#(A \backslash E) > 1$, there exist $x \neq y$ such that $\{x, y\} \subset A \backslash E$. Define the price vector r by: $r_x = p_x^0$ and $r_a = q_a(\bar{\varepsilon})$ for $a \neq x$. By GS, there exists $G \in D(r)$ such that $A \backslash \{x\} \subset G \subset A \cup E$. Now, $A \notin D(r)$ because otherwise $A \in D(q(\varepsilon))$ for some $\varepsilon < \bar{\varepsilon}$, contradicting the definition of $\bar{\varepsilon}$. Hence, $v(G, r) > v(A, r)$. But, $v(G, r) - v(A, r) = v(G, p^0) - v(A, p^0)$, so $G \in H(p^0)$, contradicting the fact that $E \in H_2(p^0)$. ∎

LEMMA 3. *If u is monotone, then SI implies NC.*

Proof. Fix a price vector p, and let $A, B \in D(p)$ and $X \subset A \backslash B$. Define

$$\mathscr{F} := \{F \in D(p) \mid F \subset A \cup B \text{ and } A \backslash X \subset F\}.$$

Note that $A \in \mathscr{F}$, so $\mathscr{F} \neq \emptyset$. Let $E \in \operatorname{argmin}\{\#(F \cap X) \mid F \in \mathscr{F}\}$. If $E \cap X = \emptyset$, we are done: define $Y := E \cap B$ and note that $E = [A, X, Y]$.

Otherwise, suppose $E \cap X \neq \emptyset$; we show that this leads to a contradiction. For each $\varepsilon \geqslant 0$, define the price vector $q(\varepsilon)$ as follows: $q_a(\varepsilon) = M > u(\Omega)$ for $a \notin A \cup B$, $q_a(\varepsilon) = p_a$ for $a \in (A \cup B) \backslash X$, and $q_a(\varepsilon) = p_a + \varepsilon$ for $a \in X$. Observe that $v(F, q(\varepsilon)) = v(F, p) - \#(F \cap X) \cdot \varepsilon$ for all $F \subset A \cup B$. Thus, for all $\varepsilon > 0$, $B \in D(q(\varepsilon))$ and $v(B, q(\varepsilon)) > v(E, q(\varepsilon))$. Hence, there exists $F \subset A \cup B$ such that $\#(E \backslash F) \leqslant 1$ and $\#(F \backslash E) \leqslant 1$, and $v(F, q(\varepsilon)) > v(E, q(\varepsilon))$ for all $\varepsilon > 0$ sufficiently small. Since D is upper semicontinuous, $F \in D(p)$. Now,

$$\begin{aligned} v(F, p) - \#(F \cap X) \cdot \varepsilon = v(F, q(\varepsilon)) &> v(E, q(\varepsilon)) \\ &= v(E, p) - \#(E \cap X) \cdot \varepsilon, \end{aligned}$$

and $v(F, p) = v(E, p)$ imply that $\#(F \cap X) < \#(E \cap X)$. Since $\#(E \backslash F) \leqslant 1$, $E \backslash X \subset F$ and $A \backslash X \subset F$. Thus, $F \in \mathscr{F}$ and $\#(F \cap X) < \#(E \cap X)$, which contradicts the definition of E. ▮

LEMMA 4. *If u is monotone, then NC implies GS.*

Proof. Let p and q be two price vectors such that $q \geqslant p$, and define

$$C := \{a \in \Omega \mid q_a > p_a\}.$$

The proof is by induction on the cardinality of C.

Suppose $\#(C) = 1$. Hence, $C = \{\alpha\}$ for some $\alpha \in \Omega$, and $q = p + (q_\alpha - p_\alpha)\, e^\alpha$. Pick ant $A \in D(p)$, and define

$$\bar{\varepsilon} := \sup\{\varepsilon \mid A \in D(p + \varepsilon e^\alpha)\}.$$

Since D is upper semicontinuous, $A \in D(p + \varepsilon e^\alpha)$ for all $\varepsilon \in [0, \bar{\varepsilon}]$. Thus, if $\bar{\varepsilon} \geqslant q_a - p_a$, then $A \in D(q)$, and we can choose $B = A$. (In particular, note that if $\alpha \notin A$, then $\bar{\varepsilon} = +\infty$.)

Suppose now that $\bar{\varepsilon} < q_a - p_a$ (so $\alpha \in A$). Note that if $\varepsilon > \bar{\varepsilon}$ and $E \in D(p + \varepsilon e^\alpha)$, then $\alpha \notin E$, for otherwise

$$0 \leqslant v(A, p) - v(E, p) = v(A, p + \varepsilon e^\alpha) - v(E, p + \varepsilon e^\alpha) \leqslant 0,$$

which contradicts the definition of $\bar{\varepsilon}$. Since Ω is finite, there exists $E \subset \Omega$ and a monotone sequence $\{\varepsilon_k\}$ of positive numbers converging to 0 such that $E \in D(p + (\bar{\varepsilon} + \varepsilon_k)\, e^\alpha)$ for all k. Again, the upper semicontinuity of D implies that $E \in D(p + \bar{\varepsilon} e^\alpha)$. Since $A \in D(p + \bar{\varepsilon} e^\alpha)$ as well, there exists $Y \subset E$ such that $[A, C, Y] \in D(p + \bar{\varepsilon} e^\alpha)$. Since $\alpha \notin [A, C, Y] =: B$, $B \in D(p + \varepsilon e^\alpha)$ for all $\varepsilon \geqslant \bar{\varepsilon}$. In particular, $B \in D(q)$, and clearly $B \supset A \backslash C$, as desired.

Suppose now that the result holds whenever $\#(C) \leqslant k$, and assume that $\#(C) = k + 1$. Pick any $\alpha \in C$ and define the price vector $\tilde{q}$ as follows: $\tilde{q}_\alpha = p_\alpha$ and $\tilde{q}_a = q_a$ for all $a \neq \alpha$. Let $\tilde{C} := C \backslash \{\alpha\}$, and pick any $A \in D(p)$. Since $\#(\tilde{C}) = k$, by inductive hypothesis, there exists $\tilde{B} \in D(\tilde{q})$ such that $\tilde{B} \supset A \backslash \tilde{C}$. By inductive hypothesis again, there exists $B \in D(q)$ such that $B \supset \tilde{B} \backslash \{\alpha\} \supset A \backslash C$. ▮

THEOREM 2. *Consider a consumer with a utility function $u_1 : \Omega \to \mathbf{R}$ that violates SI. Then, there exist $\ell - 1$ unit demand consumers with utility functions u_i, $i = 2, ..., l$, such that the economy $E = (\Omega; u_1, ..., u_\ell)$ does not have a Walrasian equilibrium.*

Proof. By assumption, there exist a price vector p and a set $A \notin D_1(p)$ such that for all $C \subset \Omega$ with $\#(A \backslash C) \leqslant 1$ and $\#(C \backslash A) \leqslant 1$ we have that

$v_1(A, p) \geqslant v_1(C, p)$. That is, A is not optimal (at prices p), but no single switch can improve A. Consider the optimization problem

$$\operatorname{argmin} \#(A \Delta C)$$
$$\text{s.t.} \quad v_1(C, p) > v_1(A, p).$$

Since $A \notin D_1(p)$, its feasible set is nonempty; let B be an optimal solution. Then, by assumption, either (i) $\# B \backslash A > 1$ or (ii) $\# A \backslash B > 1$.

Assume (i). Let $k = \# B \backslash A$ and $\varepsilon = [v_1(B, p) - v_1(A, p)]/2k > 0$.

We now introduce a collection of unit demand consumers. There is a special consumer, indexed by 2, and one consumer for each $a \notin A \cap B$. Thus, $N = \{1, 2\} \cup [\Omega \backslash (A \cap B)]$ will be the set of consumers in the economy E. Their utility functions are defined as follows. For each $a \in \Omega \backslash (A \cap B)$, let

$$u_a(C) = \begin{cases} s_a & \text{if} \quad a \in C \\ 0 & \text{otherwise,} \end{cases}$$

where

$$s_a = \begin{cases} p_a & \text{if} \quad a \in A \backslash B \\ p_a + \varepsilon & \text{if} \quad a \in B \backslash A \\ u_1(\Omega) + 1 & \text{if} \quad a \in \Omega \backslash (A \cup B). \end{cases}$$

For consumer 2, define $r_a = p_a + u_1(\Omega) + 1$ for each $a \in B \backslash A$, and

$$u_2(C) = \begin{cases} \max\{r_a \mid a \in C \cap (B \backslash A)\} & \text{if} \quad C \cap (B \backslash A) \neq \varnothing \\ 0 & \text{otherwise.} \end{cases}$$

Assume that a Walrasian equilibrium (t, Y) exists for the economy E. Since consumer 2 only values objects in $B \backslash A$, we can assume wlog that $Y_2 \subset B \backslash A$. Define q as follows: $q_a = t_a$ for $a \notin A$, $q_a = 0$ for $a \in A \cap B$, and $q_a = p_a$ for $a \in A \backslash B$. For each $a \in A \backslash B$, either $a \in Y_1$ or $a \in Y_a$. If $a \in Y_1$, then $t_a \geqslant p_a$ (since $a \notin Y_a$), and if t_a were decreased to p_a, Y_1 and Y_a would remain optimal for consumers 1 and a respectively. Similarly, if $a \in Y_a$, then $t_a \leqslant p_a$, and if t_a were increased to p_a, Y_1 and Y_a would remain optimal for consumes 1 and a respectively. Moreover, all consumers, except (possibly) consumer 1, give zero value to objects in $A \cap B$. Therefore, (q, Y) is also a Walrasian equilibrium of E. And, if we define the allocation X such that $X_1 = Y_1 \cup (A \cap B)$ and $X_i = Y_i \backslash (A \cap B)$ for $i \neq 1$, (q, X) is also a Walrasian equilibrium of E.

Clearly each agent $a \in \Omega \backslash (A \cup B)$ must consume a in equilibrium, that is $a \in X_a$ for all $a \in \Omega \backslash (A \cup B)$. Therefore

$$A \cap B \subset X_1 \subset A \cup B.$$

Agent 2 must be consuming some $a \in B \backslash A$, otherwise $q_a \geqslant p_a + u_1(\Omega) + 1$ for all $a \in B \backslash A$. But at those prices, nobody else wants to consume any $a \in B \backslash A$. Therefore, $B \backslash X_1 \neq \emptyset$.

Suppose $(B \backslash A) \cap X_1 \neq \emptyset$. Since $A \cap B \subset X_1$, we have that $\#(A \Delta X_1) < \#(A \Delta B)$, and by the minimality of B, it follows that $v_1(X_1, p) \leqslant v_1(A, p)$. For each $a \in (B \backslash A) \cap X_1$, $a \notin X_a$ and therefore $q_a \geqslant p_a + \varepsilon$. Hence, $v_1(X_1, q) < v_1(A, q)$, which is a contradiction. Therefore, $X_1 \subset A$.

We finally show that $X_1 \subset A$ also leads to a contradiction. Assume that $X_1 \subset A$. Then each $a \in B \backslash A$ is either consumed by agent 2 or agent a, and agent 2 consumes at most one object. We first show that $q_a \leqslant p_a + \varepsilon$ for all $a \in B \backslash A$. Note that if $a \in B \backslash A$ is consumed by agent a, then $q_a \leqslant p_a + \varepsilon$. Hence, if $X_2 = \emptyset$, $q_a \leqslant p_a + \varepsilon$ for all $a \in B \backslash A$. If $X_2 = \{b\}$, since $k \geqslant 2$ by assumption, there exists $a \in B \backslash A$ such that $a \neq b$. The optimality of X_2 implies that

$$p_b + u_1(\Omega) + 1 - q_b = r_b - q_b \geqslant r_a - q_a = p_a + u_1(\Omega) + 1 - q_a,$$

and since $q_a \leqslant p_a + \varepsilon$, we must have that $q_b \leqslant p_b + \varepsilon$, as desired. Now, $X_1 \subset A$ and $A \cap B \subset X_1$ imply that $\#(A \Delta X_1) < \#(A \Delta B)$, and by the minimality of B, we must have that $v_1(X_1, p) \leqslant v_1(A, p)$. But $q_a = 0$ for $a \in A \cap B$ and $q_a = p_a$ for $a \in A \backslash B$. Therefore, $v_1(X_1, q) \leqslant v_1(A, q)$. Also $q_a \leqslant p_a + \varepsilon$ for all $a \in B \backslash A$ implies that $v_1(A, q) < v_1(B, q)$. Thus, $v_1(X_1, q) < v_1(B, q)$, a contradiction.

We have shown that (i) contradicts the existence of equilibrium. Next, assume (ii) $\#(A \backslash B) > 1$ and $\#(B \backslash A) \leqslant 1$.

Now we let the set of consumers be $N = \{1, 2\} \cup [\Omega \backslash (A \cup B)]$ if $B \backslash A = \emptyset$ and $N = \{1, 2, 3\} \cup [\Omega \backslash (A \cup B)]$ if $B \backslash A$ is a singleton. Note that in the former case, $A \cap B \cap N = \emptyset$, while in case (i) before we defined N so that $A \cap B \subset N$. The utility functions of consumers $a \in [\Omega \backslash (A \cup B)]$ are defined as before. Consumer 2 now has utility function

$$u_2(C) = \begin{cases} 0 & \text{if} \quad C \cap (A \backslash B) = \emptyset \\ \max\{p_a + u_1(\Omega) + 1 \mid a \in C \cap (A \backslash B)\} & \text{otherwise.} \end{cases}$$

When $B \backslash A$ is a singleton $\{b\}$, we define

$$u_3(C) = \begin{cases} 0 & \text{if} \quad C \cap (A \Delta B) = \emptyset \\ \max\{p_a + u_1(\Omega) + 1 \mid a \in C \cap (A \Delta B)\} & \text{otherwise.} \end{cases}$$

Again by contradiction, assume that (q, X) is a Walrasian equilibrium for the economy with consumers N. As argued above, we can assume wlog that $A \cap B \subset X_1 \subset A \cup B$, and $q_a = 0$ for all $a \in A \cap B$. Finally, if $\#(X_2) > 0$, then the marginal utility of at most one object in X_2 is strictly positive for player 2. Hence all remaining objects must have 0 price and can be given to any player without upsetting the equilibrium. When agent 3 exits, the same argument can be applied to him. So, we will assume wlog that $\#(X_2) \leqslant 1$ and $\#(X_3) \leqslant 1$.

We now show that $\#(X_2) = 1$, and if agent 3 exists, then $X_3 = \{b\}$ as well. To see this note that $\#(X_2) = 0$ implies $q_a > u_1(\Omega)$ for all $a \in A \backslash B$, so agent 1 is not consuming any $a \in A \backslash B$ either, which is a contradiction. If 3 exists and does not consume b, then a similar argument yields a contradiction.

Since $A \cap B \subset X_1$ and $b \notin X_1$, we have that $v_1(X_1, p) \leqslant v_1(A, p) < v_1(B, p)$. By assumption, $\#(A \backslash B) > 1$, so $\#(X_2) = 1$ implies that $X_1 \cap (A \backslash B) \neq \varnothing$. Let

$$\varepsilon = \min\{q_a - p_a \mid a \in X_1 \cap (A \backslash B)\},$$

and c be any optimal solution of this problem. If $a \in (A \cup B) \backslash X_1$, that is, if a is consumed by player 2 (or 3), then

$$p_a + u_1(\Omega) + 1 - q_a \geqslant p_c + u_1(\Omega) + 1 - q_c.$$

Hence, $q_a \leqslant p_a - p_c + q_c \leqslant p_a + \varepsilon$. So $q_a < p_a$ for some $a \in X_1 \cap (A \backslash B)$ (i.e., $\varepsilon < 0$) implies $q_a < p_a$ for all $a \in (A \cup B) \backslash X_1$. But then $v_1(X_1, p) \leqslant v_1(A, p)$ implies $v_1(X_1, q) < v_1(A, q)$, a contradiction. Therefore, $q_a \geqslant p_a$ for all $a \in X_1 \cap (A \backslash B)$ and $\varepsilon \geqslant 0$. If $B \backslash A = \{b\}$, the optimality of $X_3 = \{b\}$ implies that

$$p_b + u_1(\Omega) + 1 - q_b \geqslant p_c + u_1(\Omega) + 1 - q_c.$$

Thus, $q_b \leqslant p_b + \varepsilon$, and $v_1(B, q) \geqslant v_1(B, p) - \varepsilon$ whether $B \backslash A$ is the empty set or the singleton $\{b\}$. Since $X_1 \cap (A \backslash B) \neq \varnothing$, $v_1(X_1, q) \leqslant v_1(X_1, p) - \varepsilon$, and therefore

$$v_1(X_1, q) \leqslant v_1(X_1, p) - \varepsilon < v_1(B, p) - \varepsilon \leqslant v_1(B, q),$$

which contradicts the optimality of X_1. ∎

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