Optimal Auctions

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In the two decades since the seminal paper by William Vickrey, literature on the theory of auctions has developed at a rapid though uneven pace.[1] Much of this literature is fragmentary, varies widely in scope, and is not easily accessible to economists. As a result, the implications of different auction rules in various settings remain relatively unknown. This paper provides a systematic examination of alternative forms of auctions. In so doing it presents a general characterization of the implications for resource allocation of different auction designs within the model originally proposed by Vickrey.

The auction model is a useful description of "thin markets" characterized by a fundamental asymmetry of market position. While the standard model of perfect competition posits buyers and sellers sufficiently numerous that no economic agent has any degree of market power, the bare bones of the auction model involves competition on only one side of the market. In this setting a single seller of an indivisible good faces a number (n) of potential buyers. Competition among the (possibly small number of) buyers takes place according to a well-defined set of auction rules calling for the submission of price offers from the buyers. Most commonly, the choice of auction method employed rests with the monopolistic seller.

These brief observations suggest two natural questions for analysis: First, what form does the competition among the few buyers take under the most common auction procedures? In turn, how is a sale price determined? Second, by what means can the seller best exploit his monopoly position? For example, would it be more profitable for the seller to require payment not only by the high bidder, but also by those with lower ranked bids?[2]

As one might expect, any change in the rules of the auction results in different bidding strategies on the part of the buyers. In particular, if the auction rules posit a minimum payment by one or more of the bidders (determined by rank), those with sufficiently low reservation values will be discouraged from entering a bid. Our analysis will demonstrate that, in a risk-neutral setting, it is the reservation value below which a buyer opts to remain out of the auction which is crucial. To be precise, if the lowest reservation value for which it is worthwhile bidding is the same for two different auction rules, then the expected return to the seller is also the same.

Throughout the paper we shall retain the following basic assumption.

a) A single seller with reservation value v_0 faces n potential buyers, where buyer i holds reservation value $v_i, i=1,\ldots,n$.

b) The reservation values of the parties are independent and identically distributed, drawn from the common distribution $F(v)$ with $F(\underline{v})=0$, $F(\bar{v})=1$ and $F(v)$ strictly increasing and differentiable over the interval $[\underline{v},\bar{v}]$. We will refer to this as the *IID* assumption.

The *IID* assumption was first presented by Vickrey, and has been frequently employed

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[1]A current bibliography by Robert Stark and Michael Rothkopf lists nearly 500 papers written over this period. For a recent survey of this literature, see Richard Engelbrecht-Wiggans.

[2]Vickrey's comparison of the open "ascending bid" auction and the sealed "high bid" auction has been generalized in unpublished dissertations by Armando Ortega-Reichert, Gerard R. Butters, and William Samuelson. Milton Harris and Artur Raviv (1979) provide the first discussion of optimal auction design. Employing very different methods they consider the special case in which each buyer has flat (uniform) prior probabilistic beliefs about the amount others are willing to pay.

in the bidding literature. In practical terms, each party is uncertain about the others' reservation values, believing that each individual decides the maximum amount he is willing to pay independently of the others. In addition, the parties share common priors with respect to the possible reservation values of each individual. With the *IID* assumption, the bidding procedures we outline below belong to the class of games of incomplete information first formulated by John Harsanyi.

The paper is organized as follows: First we present our central result demonstrating that expected seller revenue from quite different auctions can be very easily compared. As an immediate implication, the equivalence of the "English" or "ascending bid" auction and the "Dutch" or "high bid" auction is established. More important, it is shown that, for a broad family of auction rules, expected seller revenue is maximized using either of the two common auctions if the seller announces that he will not accept bids below some appropriately chosen minimum or "reserve" price. Surprisingly, this reserve price is independent of the number of buyers and is always strictly greater than the seller's personal value of the object. In Section II several alternative auction rules are described in detail and their implications for the seller are compared. Finally, in Section III, the two commonly used auctions are once again compared under the assumption that the buyers are risk averse rather than risk neutral. It is shown in this setting that the English auction is dominated by the sealed high bid auction, and that the optimal reserve price in the latter is a declining function of the degree of buyer risk aversion.

I. Comparison of Alternative Auction Rules

Before characterizing a broad family of alternative auction rules, a few remarks about the English or ascending bid auction will be helpful.

In auctions of antiques, estate objects, and works of art, the good is awarded to the buyer who makes the final and highest bid. The buyer placing the highest valuation on the good therefore pays approximately the maximum of the reservation values of the other $n-1$ buyers. As Vickrey noted, this is equivalent to a sealed bid auction in which each buyer submits a bid and the high bidder pays the *second* highest rather than the highest bid.[3] To see this, suppose the ith buyer considers shading his bid, b_i, below his reservation value v_i. If the largest of all the other bids, b_*, exceeds v_i, another buyer is the high bidder so that buyer i's gain remains zero. If $b_* < b_i$, buyer i remains the high bidder and continues to gain $v_i - b_*$. However, if $b_i < b_* < v_i$, the shading yields a zero gain, whereas without shading the gain is $v_i - b_*$. A parallel argument establishes that there is no advantage in making a bid, b_i, greater than v_i. The optimal strategy of each buyer is therefore to submit his reservation value. It follows that, just as in the English auction, the high bidder ends up paying the second highest reservation value. This equivalence greatly simplifies the comparison between the English and sealed high bid auctions[4] since it implies that we need only compare sealed bid auctions.

In the high and second bid auctions, only the winner makes a payment to the seller. However, there is an infinity of auction rules involving payment by more than one bidder. For example, all buyers might be charged a fixed entry fee. Alternatively, losers might be required to pay some fraction of their bids. A third possibility, discussed in Section II, is that the seller might attempt to encourage higher bids by offering to return some of the money paid by the winner to each of the losers, the size of the rebate depending on a loser's bid.

Each of these alternatives is an example of an auction with the following properties. First, a buyer can make any bid above some minimum "reserve" price announced by the seller. Second, the buyer making the highest bid is awarded the object. Third, the auction rules are anonymous: each buyer is treated

[3] This type of auction is sometimes referred to as a Vickrey auction.

[4] The sealed high bid auction also has its open auction equivalent. In this Dutch auction, the sale price is initially set at a high level and is then lowered until a bid is made.

alike. Fourth, there is a common equilibrium bidding strategy in which each buyer makes a bid b_i, which is a strictly increasing function of his reservation value v_i, i.e.,

$$b_i = b(v_i) \quad i=1,\dots,n \tag{1}$$

Throughout this section we shall consider the family $\mathcal{Q}$ of auction rules for which these four assumptions are satisfied.[5] We begin with the main result.

PROPOSITION 1: *Suppose the IID assumption holds and all buyers are risk neutral. The common equilibrium bidding strategy for any member of the family $\mathcal{Q}$ of auction rules yields an expected revenue to the seller of*

$$n\int_{v_*}^{\bar{v}} (vF'(v)+F(v)-1)F(v)^{n-1}\,dv$$

where v_* is the reservation value below which it is unprofitable to submit a bid.

Proposition 1 is important because it tells us that the expected revenue of the seller from quite different auctions can be compared simply by determining the lowest reservation value, v_*, for which it is worthwhile bidding. We begin the proof by examining the behavior of a single buyer. The expected return to making a bid can be expressed as follows.

$$\begin{Bmatrix}\text{expected}\\ \text{buyer}\\ \text{gain}\end{Bmatrix} = \begin{Bmatrix}\text{reservation}\\ \text{value}\end{Bmatrix}\begin{Bmatrix}\text{probability}\\ \text{of}\\ \text{winning}\end{Bmatrix} - \begin{Bmatrix}\text{expected}\\ \text{payment}\end{Bmatrix} \tag{2}$$

Below we obtain simple expressions for both the probability of winning and the expected buyer gain. Then, from (2), we are able to derive the expected payment of a typical buyer. Given the symmetry of the auction rule, expected seller revenue is just n times this expected payment.

With buyers behaving noncooperatively, a common strategy, $b_i = b(v_i)$, is an equilibrium strategy if, when adopted by all buyers but one, the latter's best response is to adopt it also. Without loss of generality, we may suppose that the buyer considering an alternative bidding strategy is buyer 1. With all other buyers bidding according to $b(v)$, buyer 1, if he bids at all, will wish to bid in the range of this function. Hence, we can write any bid as $b_1 = b(x)$ and view buyer 1 as choosing x. It follows that $b(v)$ is an equilibrium bidding strategy if buyer 1 can do no better than choose $x = v_1$, and so bids $b(v_1)$.

To examine the optimal choice of buyer 1, we begin by assuming his bid is $b(x)$, and then ask what restrictions are implied by the requirement that his optimal bid is $b(v_1)$. Any auction rule must specify the amount he must pay, p, given his own bid $b_1 = b(x)$ and those by the other $n-1$ buyers, i.e.,

$$\begin{aligned} p &= p(b_1, b_2, \dots, b_n) \\ &= p(b(x), b(v_2), \dots, b(v_n)) \end{aligned}$$

We may therefore write the expected payment by buyer 1, given a bid of $b_1 = b(x)$ as

$$P(x) = \underset{v_2,\dots,v_n}{E}\, p(b(x), b(v_2), \dots, b(v_n)) \tag{3}$$

Also, the bid of $b(x)$ is the winning bid if and only if all other buyers have made lower bids. By assumption the equilibrium bid function is strictly increasing in v, therefore buyer 1 wins if all other valuations are less than x. Since the probability buyer j has a reservation value less than x is $F(x)$, buyer 1 wins with probability $F^{n-1}(x)$. Combining this last result with (2) and (3), the expected gain to buyer 1, if he chooses to enter the auction, can be expressed as

$$\Pi(x, v_1) = v_1 F^{n-1}(x) - P(x) \tag{4}$$

[5]After deriving Proposition 1, we became aware of a paper by Roger Myerson which uses a much more technically demanding approach to examine expected seller revenue in an even broader class of auctions. Generalizing our approach, Eric Maskin and Riley (1980a) have shown that Myerson's results imply there there are circumstances in which expected seller revenue can be increased by prohibiting bids over certain ranges.

For $b(v)$ to be the equilibrium bidding strategy, buyer 1's optimal choice must be to select $x=v_1$ and bid $b(v_1)$. Then buyer 1's maximized expected gain is $\Pi(v_1, v_1)$ and the following first-order condition must be satisfied.[6]

$$(5)\quad \frac{\partial \Pi}{\partial x}(x, v_1) = v_1 \frac{d}{dx}F^{n-1}(x) - P'(x) = 0 \quad \text{at } x=v_1$$

This must hold for all reservation values exceeding v_*, the reservation value for which a buyer is indifferent between submitting the bid, $b(v_*)$, and not entering the auction. That is, (5) holds for all $v_1 \geqslant v_*$, where v_*, satisfies

$$(6)\quad \Pi(v_*, v_*) = v_* F^{n-1}(v_*) - P(v_*) = 0$$

Setting $x=v_1$ in (5), it follows that the equilibrium expected payment by buyer 1 must satisfy the differential equation

$$(7)\quad P'(v_1) = v_1 \frac{d}{dv_1} F^{n-1}(v_1) \qquad v_1 \geqslant v_*$$

Integrating and making use of the boundary condition, (6), buyer 1's expected payment is therefore

$$(8a)\quad P(v_1) = v_* F^{n-1}(v_*) + \int_{v_*}^{v_1} x\, dF^{n-1}(x) \qquad v_1 \geqslant v_*$$

Integrating the second term by parts, this can be rewritten more conveniently as

$$(8b)\quad P(v_1) = v_1 F^{n-1}(v_1) - \int_{v_*}^{v_1} F^{n-1}(x)\, dx \qquad v_1 \geqslant v_*$$

[6]From (4) and (5) we also have

$$\frac{\partial \Pi}{\partial x}(x, v_1) = (v_1 - x)(n-1)F^{n-2}(x)F'(x)$$

Therefore $\Pi(x, v_1)$ is increasing in x for $x<v_1$ and decreasing for $x>v_1$ and the first-order condition yields the global maximum for buyer 1.

The final step is to consider the auction from the seller's viewpoint. As far as the seller is concerned, v_1 and hence the expected payment, $P(v_1)$, is a random variable. The seller's expected revenue from buyer 1 is therefore the expectation of $P(v_1)$. Since the seller knows that v_1 has distribution $F(v_1)$ his expected revenue is

$$\bar{p}^1 = \int_{v_*}^{\bar{v}} P(v_1) F'(v_1)\, dv_1$$

Substituting for $P(v_1)$ from (8b) and integrating by parts, the expected revenue from buyer 1 can be rewritten as follows:

$$(9)\quad \bar{p}^1 = \int_{v_*}^{\bar{v}} \left[vF'(v) + F(v) - 1 \right] F^{n-1}(v)\, dv$$

Given the equal treatment of all n buyers, expected seller revenue is just n times the expected revenue from buyer 1 and the proposition is proved.

One of the striking features of our derivation is that nowhere is there explicit reference to the equilibrium bidding strategy $b(v)$. However, under any particular auction rule this is readily derived. The key to such derivation is (8), the expression for expected payment, $P(v)$, of a buyer with reservation value v.

For example, in the high bid auction, suppose the seller announces a reserve price b_0. Any buyer with a reservation value $v>b_0$ has an incentive to enter, i.e., $v_* = b_0$. Since a buyer pays if and only if he is the high bidder, his expected payment is

$$(10)\quad P(v) = \mathit{Prob}\,\{b(v) \text{ is high bid}\}\; b(v)$$

But $b(v)$ is the high bid if and only if all other buyers have lower reservation values. Then *Prob* $\{b(v)$ is high bid$\} = F^{n-1}(v)$ and, from (10), $b(v) = P(v)/F^{n-1}(v)$. Substituting for $P(v)$ from (8b), we therefore have the following additional result.

PROPOSITION 2: *Suppose the IID assumption holds, all buyers are risk neutral and the seller announces a reserve price b_0. In the high* *bid auction, the equilibrium bidding strategy of a typical buyer with reservation value $v \geqslant b_0$ is*

$$b(v)=v-\int_{b_0}^{v} F^{n-1}(x)dx/F^{n-1}(v)$$

Proposition 2 indicates the degree to which a buyer will "shade" his bid, $b(v)$, below his reservation value v in the high bid auction. It is a straightforward matter to confirm that $b(v)$ is strictly increasing in v. Therefore the high bid auction is a member of the family of auctions described by Proposition 1. Certainly the second bid auction is in this family since anyone entering will bid his reservation value. In both auctions, a buyer will enter if and only if his reservation value exceeds b_0. Then in both cases $v_*=b_0$ and, from Proposition 1, expected seller revenue is the same.

We now demonstrate that these two common auction rules are optimal for the seller, given the appropriate choice of a reserve price. For any auction rule there is some implied minimum reservation value v_* below which buyers will choose not to bid. Then there is a probability of $F^n(v_*)$ that all n buyers will decide not to submit a bid. In this case the seller's gain is his own personal valuation v_0. Then, from (9), the total expected return to the seller is

$$v_0F^n(v_*)+n\int_{v_*}^{\bar{v}}(vF'(v)+F(v)-1)F^{n-1}(v)dv \tag{11}$$

It follows that any two auctions in the family $\mathcal{Q}$, for which v_* is the same, yield the same expected gain to the seller.[7] Moreover, differentiating with respect to v_* the expected gain of the seller is maximized for some v_* satisfying the condition[8]

$$n\left[v_0F'(v_*)-v_*F'(v_*)-F(v_*)+1\right]F^{n-1}(v_*)=0 \tag{12}$$

[7]Moreover, any two auctions for which v_* is the same yield the same expected gain to buyer i conditional on v_i. This result follows directly from equations (4) and (8b).

[8]Expression (12) will, in general, have multiple roots. If this is the case, it is necessary to evaluate the expected return, (11), at each root to determine the global maximum.

We therefore have the following further result.

PROPOSITION 3: *If the IID assumption holds and buyers are risk neutral, the members of the family $\mathcal{Q}$ of auction rules, which maximize the expected gain of the seller are those for which the reservation value v_*, below which it is not worthwhile bidding, satisfies*

$$v_*=v_0+1-F(v_*)/F'(v_*)$$

independent of the number of buyers.

An immediate implication of Proposition 3 is that the high and second bid auctions with reserve price $b_0=v_*$ are both optimal in the family of auctions $\mathcal{Q}$. Note also that v_* exceeds v_0: the seller announces a reserve price strictly greater than his personal valuation.

To gain an understanding of this strong result, it is helpful to consider a second bid auction in which there are two buyers and to examine the implications of introducing a reserve price slightly higher than v_0; i.e., $v_*=v_0+\delta$ where δ is small. Since each buyer's dominant strategy is to bid his reservation value, the expected gain to the seller is affected (i) if both valuations lie between v_0 and v_*, and (ii) if one valuation lies between v_0 and v_* and the other exceeds v_*. In the first case the seller retains the item, which he values at v_0, rather than selling it at some price between v_0 and v_*. His loss is therefore of order δ. Since this outcome occurs with probability $(F(v_*)-F(v_0))^2 \approx (F'(v_0))^2\delta^2$, the expected loss is of order δ^3. In the second case the seller receives a payment of v_* rather than some price between v_0 and v_* hence has a gain of order δ. Since this outcome occurs with probability $2(F(v_*)-F(v_0))(1-F(v_*)\approx 2F'(v_0)(1-F(v_0)\delta$, the expected gain is of order δ^2. Therefore, for sufficiently small δ, the gain to raising the reserve price above the seller's personal valuation outweighs the cost.

While we have focused on the reserve price because of its common usage, there are many different ways in which to discourage the appropriate subset of buyers from participating in the bidding. Suppose, for example, that the seller announces a fixed entry fee c. For all buyers with valuations less than some number v_c, it will be optimal to remain out of the auction. Consider a buyer with the borderline reservation value v_c. In the second bid auction he enters, and, since the entry fee is now sunk, bids his true value v_c. He wins if and only if there are no other bidders, in which case there is no additional payment. Since this occurs with probability $F(v_c)^{n-1}$ his expected profit is

$$v_c F(v_c)^{n-1} - c \tag{13}$$

But for v_c to be the borderline reservation value, the expected profit must be zero. The seller then chooses an entry fee c_* satisfying

$$c_* = v_* F(v_*)^{n-1} \tag{14}$$

A similar argument holds for the high bid auction. If a buyer has the borderline reservation value v_c, he wins if and only if there are no bidders. The optimal bid in such circumstances is zero, therefore the expected profit is again given by (13) and the optimal entry fee by (14).

Our general results are also helpful in analyzing the expected payoff to multiple rounds of bidding. Suppose, for concreteness, that the seller is using a high bid auction. If he can convince buyers that there will only be a single round with optimal reserve price v_*, there will be a chance that no bids will be submitted. Since v_* exceeds the seller's reservation value v_0, the seller, after the fact, has an incentive to lower his reserve price and to call for a second round of bids. However, if buyers are not fooled, they will adopt first-round strategies in the expectation of a possible second round. In particular, all those with reservation values $v_j > b_{00}$, the reserve price in the second round, will plan to enter the two round auction. Then, from Proposition 3 the seller's expected gain is lower since the entry value is no longer optimal.

A final point concerns the decision of the seller whether or not to announce a reserve price. In the second bid auction, the strategy of bidding one's reservation value is a dominant strategy. Therefore the seller cannot influence bids by concealing his reserve price. It follows that the optimal silent reserve price is the same as the optimal announced reserve price and that expected seller revenue is identical.

The argument is more complex in the case of the high bid auction, but once again it can be shown that there is no advantage in using a silent reserve price. The proof, which involves a straightforward extension of Proposition 1, is provided in our earlier paper.

II. Alternative Auctions

To illustrate the results of Section I, we now compare some specific auctions under the simplifying assumptions that there are only two buyers, and that reservation values are uniformly distributed on the unit interval ($F(v)=v$, for $v \in [0,1]$). We assume also that the object for sale has no value to the seller, $v_0 = 0$. First we indicate the gains to employing an optimal reserve price in the high bid auction. We then present an unusual pair of auction designs which happen to belong to the class of optimal auctions.[9] In contrast, a third example shows that a seemingly natural (and commonly employed) auction procedure is suboptimal.

Under our simplifying assumptions, it follows from Proposition 3 that it is optimal for the seller to design the auction so that only those with reservation values exceeding $v_* = 1/2$ find it worthwhile bidding. Then, in the high bid auction it is optimal for the seller to announce a minimum or reserve price $b_0 = 1/2$. Appealing to Proposition 2, this, in turn, implies that the equilibrium bid of buyer i, with reservation value $v_i \geqslant 1/2$, is $b(v_i) = v_i/2 + 1/8v_i$. By contrast, if the seller always sells the good (by setting the reserve price $b_0 = 0$) buyer i's bid becomes $\tilde{b}(v_i) = v_i/2$. Either by direct computation or by appealing to Proposition 1, it can be confirmed that expected seller revenue is $5/12$ with $b_0 = 1/2$ and $1/3$ with $b_0 = 0$. Thus the optimal re-

[9] The interested reader is referred to our earlier paper, where it is shown that the auctions of examples 1 and 2 belong to the class of optimal auctions for arbitrary $F(v)$ and n.

serve price strategy results in a 25 percent increase in expected revenue.

We now consider three quite different auction rules.

Example 1: *Sad Loser Auction*. Suppose there are just two buyers and the seller announces the following auction rules.

(i) Each buyer paying an entry fee c is eligible to submit as his bid any positive real number.[10]

(ii) The high bidder receives the good but retains his bid.

(iii) The lower bidder (if there is one) loses his bid.

It is tempting to conjecture that there is no equilibrium bidding strategy for this set of rules. However, not only is this incorrect, but the equilibrium bidding strategy is readily derived. Under rules (i)–(iii), the expected gain of the typical buyer is

$$\Pi(x, v_i) = v_i F(x) - b(x)(1 - F(x))$$

For $b(v)$ to be the equilibrium bidding strategy, this gain is maximized by setting $x = v_i$ so that the buyer's expected payment is $b(v_i)(1 - F(v_i))$. Then if $F(v) = v$ and $c = 1/4$, it can be confirmed from equation (14) that $v_* = 1/2$, and from equation (8b) that

$$b(v_i) = \frac{(v_i^2 - 1/4)}{2(1 - v_i)} \qquad \text{for } v_i \geqslant \frac{1}{2}$$

Thus, as in the optimal high bid auction, any buyer with reservation value less than $1/2$ remains out of the auction. The bids of those with higher reservation values are strictly increasing in v and increase without bound as v approaches 1! Nevertheless, it is easy to confirm that expected seller revenue under this scheme matches that of the high bid and second bid auction, cum optimal reserve price.

Under the high bid and second bid auctions, only the recipient of the good gains. In contrast, the following auction distributes a positive return to all participants.

Example 2: *Santa Claus Auction*. Suppose there are just two buyers, and the seller announces the following auction rules.

(i) A buyer who submits a bid $b \geqslant v_*$ receives from the seller an amount $S(b) = \int_{v_*}^{b} F(v)\,dv$.

(ii) The high bidder obtains the good for his bid price so that his net payment is $b - S(b)$.

One can confirm that the equilibrium strategy of each buyer is to bid his reservation value. Suppose that the second buyer bids $b_2 = v_2$. Then if buyer 1 bids b_1, his expected profit is given by

$$\begin{aligned} Pr\{b_1 \text{ is high bid}\}(v_1 - b_1) + S(b_1) \\ = F(b_1)(v_1 - b_1) + S(b_1) \end{aligned}$$

It is straightforward to check that this expression is maximized at $b_1 = v_1$.

In this auction the seller's expected net revenue is the expected value of the higher of the two bids less the seller's expected payments. With $v_* = 1/2$, $S(b) = b^2/2 - 1/8$. Moreover, each buyer bids his reservation value; therefore the seller's expected gross receipts and payments are easily computed. Once again it can be confirmed that expected net revenue is $5/12$, exactly the sum the seller can expect from the high bid auction.

Since the implication of Proposition 1 is that many seemingly different auction techniques lead to the same ultimate results, it is important to illustrate the range of exceptions.

Example 3: *Matching Auction*. Suppose there are just two buyers and the seller employs the following auction rules.

(i) There is a single round of bidding. Buyer 1 is given the opportunity to quote a price $b_1 \geqslant v_*$.

(ii) If buyer 1 makes a bid, buyer 2 can match it, if he chooses, obtaining the good for this price. If buyer 1 makes no bid, buyer 2 can obtain the good at price v_* if he chooses.[11]

[10] This rules out bids such as "infinity" or "one more than my opponent."

[11] For an analysis of the matching auction when m rounds of bidding are permitted, see our earlier paper.

Though this auction procedure is quite common (for example, in house sales, a renter occupant is frequently given the right to match the offer of any potential buyer), it is inefficient from the point of view of the seller. In fact, in some circumstances it permits a buyer who values the item less highly than his opponent to obtain the good. Thus, it may produce an allocation of the good that is inefficient *ex post*.

Suppose that $F(v)=v$ and $v_*=1/2$. The strategy of buyer 2 is straightforward. He matches b, if and only if $v_2 \geqslant b$. If buyer 1 does not open the bidding, buyer 2 bids 1/2 for the good if $v_2 \geqslant 1/2$. Anticipating the behavior of buyer 2, buyer 1 bids $b_1 \geqslant 1/2$ to maximize

$$(v_1 - b_1)\,Prob\{\text{buyer 2 chooses not to match}\}$$

Buyer 2 will not match if his reservation value is less than b_1, that is, he will not match with probability b_1. Then buyer 1 chooses $b_1 \geqslant 1/2$ to maximize his expected gain $(v_1 - b_1)b_1$. Since this expression is decreasing in b_1 for all $b_1 > 1/2$, buyer 1's optimal strategy is to bid

$$b_1(v)=\begin{cases} 0 & v_1 < 1/2 \\ 1/2 & v_1 \geqslant 1/2 \end{cases}$$

Consequently, whenever $1/2 < v_2 < v_1$, the object is awarded to buyer 2 who values it less highly than buyer 1. The expected revenue of the seller for this example is 3/8, a reduction of 10 percent relative to the high and second bid auctions.

III. Buyer Risk Aversion

When potential buyers are risk averse, the fundamental equivalence result outlined in Section I is no longer valid.[12] Retaining the assumption of buyer symmetry, it is shown in the Appendix that the high bid auction dominates the second bid auction under buyer risk aversion.

PROPOSITION 4: *Suppose assumption IID holds and all buyers share a common utility function displaying risk aversion. Then* (i) *In the second bid auction, bidders continue to bid their reservation values, that is,* $b_i = v_i$. (ii) *In the high bid auction, as bidders become more risk averse, they make uniformly higher bids.* (iii) *Consequently, the seller enjoys a greater expected profit under the high bid auction than under the second bid auction.*

It is evident that the introduction of risk aversion does not affect the strategy dominance of bidding one's true reservation value in a second bid auction, hence part (i). Part (iii) follows directly from part (ii) which is proved in the Appendix.

The intuition behind these results is that with risk aversion the marginal increment in wealth associated with a successful, slightly lower bid is weighted less heavily than the possible loss $(v_i - b_i)$ if, as a result of lowering the bid, the buyer is no longer the high bidder. This leads risk-averse bidders always to shade their bids less than risk-neutral bidders.

Under risk aversion, the general equivalence result obtained in Proposition 1 no longer holds. For instance, an auction employing a seller reserve price will not, in general, be equivalent to one that specifies a buyer entry fee—even when the same reservation value v_*, below which it is not worth bidding, is implied. Still it is natural to explore the effect that buyer risk aversion has on the optimal seller reserve price in the high bid auction. The following result is derived in the Appendix.

PROPOSITION 5: *Suppose assumption IID holds and all buyers share a common cardinal utility function. Then, in the high bid auction, the optimal seller reserve price is a declining function of the degree of risk aversion.*

The proposition is intuitively plausible in view of the fact that as buyers become risk

[12] Other authors have also considered the effects of risk aversion on bidding. Butters derives Propositions 4 and 5 for the special case in which buyers exhibit constant relative risk aversion. Charles Holt examines the effects of risk aversion in the closely related problem of bidding on incentive contracts. Steven Matthews compares high bid and second bid auctions when seller and buyers are risk averse.

averse in the extreme, the amount by which they will shade their reservation values approaches zero, $b(v_i) \to v_i$. Naturally, the seller can do no better than to announce his personal valuation as his reserve price, $b_0 = v_0$. To quote a higher price cannot "push up" buyer offers and risks the loss of beneficial sales. Of course when $b_0 = v_0$ and $b_i = v_i$, the high bid auction is also efficient *ex post*.

IV. Concluding Remarks

While a general result concerning the design of optimal auctions under uncertainty has been presented, it is important to point out the limitations and special assumptions of the present model. We have assumed that:

(a) A single indivisible good is to be sold to the highest bidder.

(b) The greater a bidder's reservation value the more he will bid for the good.

(c) Buyer roles are symmetrical (i.e., buyer values are drawn from a common distribution) and each buyer is risk neutral.

(d) Buyer values are independent.

Additional difficulties are raised when multiple goods are auctioned or when a divisible good must be allocated. Unless buyer valuations are additive and income independent, auctioning the goods in sequence will be inefficient (*ex post* and *ex ante*). When multiple goods are auctioned, each buyer should logically submit a bid for each subset of goods. Roughly speaking, the seller will allocate goods to maximize revenue under one of a number of auction schemes. In the case of a divisible good, each buyer will submit a "demand schedule" indicating the price he is willing to pay for any given quantity of the good. The seller must formulate an auction rule which specifies the allocation of the good and appropriate payment of buyers. In either instance the determination of optimal auctions for these more general environments lies beyond the bounds of the present analysis.[13]

Given assumption (b) it follows that the family of auctions considered are those in which the good is sold to the buyer with the highest reservation value v, if the good is sold at all. Under only moderate restrictions on the form of the distribution $F(v)$, it can be shown that it is never optimal to utilize a rule in which the winner might be someone other than the buyer with the highest v. However, when these restrictions are not satisfied, a stochastic auction is optimal. In such an auction a lottery is employed to allocate the good when buyer reservation values fall in specified ranges.[14]

Dropping the assumption of buyer symmetry also causes complications in the analysis. The derivation of the class of optimal auctions relied explicitly on the existence of a *common* equilibrium bidding strategy. Without this, these propositions no longer hold. The asymmetric model, though far more complex, is nevertheless amenable to the basic approach developed herein. Suppose the reservation prices of the buyers are drawn from the independent distributions, $F_1, F_2, \ldots, F_n$. Some partial results from this setting suggest a basic conclusion. An optimal auction extends the asymmetry of the buyer roles to the allocation rule itself. The assignment of the good and the appropriate buyer payment will depend not only on the list of offers, but also on the identities of the buyers who submit the bids. In short, an optimal auction under asymmetric conditions violates the principle of buyer anonymity.

As pointed out earlier, the assumption of risk neutrality is crucial to our general equivalence result. Given risk neutrality, the seller can do no better than to employ the second bid auction with an optimal reserve price. In this auction, buyers will have no difficulty formulating an optimal bidding strategy. Nor need they know the form of the distribution function $F(v)$. Against any distribution of opponents' bids, each buyer's dominant strategy is to bid his reservation value. The clear advantage of the second bid auction is that it economizes on the information each buyer requires to bid optimally. Furthermore, Proposition 3 indicates that the seller

[13]Harris and Raviv (1981) and Maskin and Riley (1980b) analyze optimal auctions for different classes of demand curves.

[14]For a presentation of the more general framework from which the optimal stochastic auction can be derived see Myerson or Maskin and Riley (1980a).

can formulate an optimal reserve price policy without knowledge of the number of buyers who might enter the auction.[15]

Finally, one must consider the appropriateness of the model's most basic assumption, value independence. The analysis has assumed that each buyer is informed of his own reservation price and, more important, that this price conveys no information about any other buyer's value. A different auction model has been applied to bidding for offshore oil leases.[16] Here, a tract being auctioned is assumed to have a common value for all parties. The tract value is unknown, though buyers may possess (differing) sample information allowing inferences about this value. In this setting, each buyer must determine a strategy for acquiring information concerning the value of the tract and for submitting a bid based on a correct estimate of this value. These features have a direct influence on the determination of an optimal auction and raise additional policy issues. (Should the seller maintain a stake in an awarded tract for the purpose of risk sharing? Should the seller undertake measures to facilitate information acquisition or to allow information pooling?)[17]

In most real world settings, we would expect that a good's economic value to a potential buyer consists of two parts—a value element which is common to all market participants and one which is buyer specific. Shell's recent \$3.6 billion purchase of Belridge Oil—the most expensive in *U.S.* history—is a dramatic example. Belridge was sold by closed sealed bid auction in which twenty-odd prospective buyers participated. Differences in bids presumably reflected (i) differences in beliefs about the value of Belridge's oil holdings (differences that might have been dissipated through pooling of information) and (ii) differences in the extent to which Belridge's operations complemented bidders' other activities. It is easy to imagine, though not to solve, a hybrid model specifying both dependent and independent components of buyer reservation values. A formal analysis of optimal auction design in this more general environment remains to be undertaken.

APPENDIX

PROPOSITION 4 (ii): *Suppose assumption IID holds and all buyers share a common utility function displaying risk aversion. Then in the high bid auction, as bidders become more risk averse, they make uniformly higher bids.*

PROOF:

Let $b(v)$ be the common equilibrium strategy of n risk averse buyers, each of whom has the same von Neumann-Morgenstern utility function $u(x)$. We assume that $u(x)$ is a strictly increasing, concave function of x and normalize so that $u(0)=0$. With all other buyers using the equilibrium bidding strategy and buyer j bidding $b(x)$, j's expected utility is

$$F^{n-1}(x)u(v_j-b(x)) \tag{A1}$$

For $b(x)$ to be the equilibrium strategy, (A1) must have its maximum at $x=v_j$. Differentiating with respect to x and setting the derivative equal to zero at $x=v_j$, we have the

[15] The largest auction houses (for example, Sotheby Park Bernet, Inc. and Christie's) employ the English auction (combining its open bid and sealed bid forms) to sell rare and valuable items (art, antiques, and jewelry). A buyer can bid personally for an item on the day of the auction or can submit a prior written offer, designating a representative from the auction house to bid on his behalf. This same procedure establishes a silent seller reserve price, since a house representative is instructed to buy back the good if the sale price is insufficient. It is a common observation that the competitive features of the open ascending auction serve to elevate buyer offers (above their prior values). This implies that the open ascending auction enjoys a practical advantage over the sealed bid version. The "mixed" auction allows written bids in order to promote the greatest possible participation while maintaining the "uplifting" features of the open ascending auction.

[16] See, for example, Robert Wilson (1975) and Matthew Oren and Albert Williams. In this model buyers begin with common prior beliefs about the value of a resource but have different *posterior* beliefs as a result of independent sampling. For discussion of auctions in which buyers have different prior beliefs, see Wilson (1967).

[17] For a discussion of the incentives for the seller to make information public, see Paul Milgrom and Robert Weber.

necessary condition

$$(n-1)F^{n-2}(v_j)F'(v_j)u(v_j-b(v_j)) - F^{n-1}(v_j)u'(v_j-b(v_j))\frac{db}{dv_j}=0$$

Rearranging yields the following differential equation for $b(v)$

$$\text{(A2)} \quad b'(v)=(n-1)\frac{F'(v)}{F(v)}\frac{u(v-b)}{u'(v-b)}$$

With reserve price $b_0=v_*$ we also have the boundary condition

$$\text{(A3)} \qquad b(v_*)=v_*$$

We wish to compare the solution for two different utility functions, $u_1(\cdot)$ and $u_2(\cdot)$ where the latter exhibits a higher degree of risk aversion, that is,

$$\text{(A4)} \quad -u_2''(x)/u_2'(x)>-u_1''(x)/u_1'(x)\geqslant 0$$

By inspection of (A2), if we can establish that

$$\text{(A5)} \qquad \phi(x)=u_2(x)/u_2'(x) - u_1(x)/u_1'(x)>0 \qquad \text{for } x>0$$

then $b_2'(v)>b_1'(v)$ and hence $b_2(v)>b_1(v)$ for all $v>v_*$. To demonstrate (A5) we note first that, since $u(0)=0$ and $u(x)$ is strictly increasing,

$$\text{(A6)} \qquad \frac{u(x)}{u'(x)}>\frac{u(0)}{u'(0)}=0 \qquad \text{for all } x>0$$

Inequality (A5) holds if we can establish that for all x such that $\phi(x)=0, \phi(x)$ is strictly increasing. Differentiating (A5) we have

$$\text{(A7)} \quad \phi'(x)=\left(\frac{-u_2''}{u_2'}\right)\left(\frac{u_2}{u_2'}\right)-\left(\frac{-u_1''}{u_1'}\right)\left(\frac{u_1}{u_1'}\right)$$

From (A4)–(A6), $x>0$ and $\phi(x)=0$ implies that $\phi'(x)>0$. Moreover, differentiating (A7) and setting $x=0$ we also have

$$\phi''(0)>\phi'(0)=0$$

Thus $\phi(x)$ is strictly increasing at $x=0$.

PROPOSITION 5: *Suppose assumption IID holds and all buyers share a common von Neumann-Morgenstern utility function. Then in the high bid auction, the optimal seller reserve price is a declining function of the degree of risk aversion.*

PROOF:

The method of proof is to compare the effect of a change in the reserve price v_* on the equilibrium bid function $b=b(v, v_*)$ for different degrees of risk aversion. Expected seller revenue, $R(v_*)$, is the expected value of the highest ranked bid, that is,

$$R(v_*)=\int_{v_*}^{\bar{v}} b(v, v_*)\,dF^{n-1}(v)$$

Then the net advantage to the seller if utility is $u_2(\cdot)$ rather than $u_1(\cdot)$ can be expressed as

$$R_2(v_*)-R_1(v_*) = \int_{v_*}^{\bar{v}}\left[b_2(v, v_*)-b_1(v, v_*)\right]dF^{n-1}(v)$$

Differentiating with respect to v_* we have

$$\text{(A8)} \quad R_2'(v_*)-R_1'(v_*) = \int_{v_*}^{\bar{v}}\left[\frac{\partial b_2}{\partial v_*}-\frac{\partial b_1}{\partial v_*}\right]dF^{n-1}(v)$$

It suffices to show that the bracketed expression in (A8) is negative, for then $R_2'(v_*)$ is negative when $R_1'(v_*)$ is zero.

From (A2), the equilibrium bid function $b(v,v_*)$ is the solution to

$$\text{(A9)} \quad \frac{\partial}{\partial v}b(v, v_*)=(n-1)\frac{F'(v)}{F(v)}\frac{u(v-b)}{u'(v-b)}$$

with the boundary condition,

$$\text{(A10)} \qquad b(v_*, v_*)=v_*$$

Assuming $u(\cdot)$ is twice differentiable, we can differentiate (A9) with respect to the reserve price v_* and so obtain the following differential equation for $\partial b/\partial v_*$

$$\text{(A11)} \quad \frac{\partial}{\partial v}\left(\frac{\partial b}{\partial v_*}\right)=-(n-1)\frac{F'(v)}{F(v)} \times\left[1+\left(\frac{-u''}{u'}\right)\left(\frac{u}{u'}\right)\right]\left(\frac{\partial b}{\partial v_*}\right)$$

From (A4) and (A5) the bracket in (A11) is larger for the utility function $u_2(x)$ exhibiting greater risk aversion. Then if we can establish that $\partial b_2/\partial v_* = \partial b_1/\partial v_* > 0$ at $v = v_*$, it will follow from (A11) that

$$\frac{\partial}{\partial v}\left(\frac{\partial b_2}{\partial v_*}\right) > \frac{\partial}{\partial v}\left(\frac{\partial b_1}{\partial v_*}\right)$$

for $v > v_*$ and hence that $\partial b_2/\partial v_* > \partial b_1/\partial v_*$ for $v > v_*$.

From (A10) we have,

$$\text{(A12)} \quad \frac{\partial b}{\partial v}(v, v_*)|_{v=v_*} + \frac{\partial b}{\partial v_*}(v, v_*)|_{v=v_*} = 1$$

Since $b(v_*, v_*) = v_*$ and $u(0)=0$, it follows from (A2) that for any concave utility function and any $v_* > 0$, the first term in (A12) is zero. Then the second term in (A12) is equal to unity for both $u_1(x)$ and $u_2(x)$.

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