Games and Economic Behavior 108 (2018) 225–238

Contents lists available at ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

From the bankruptcy problem and its Concede-and-Divide solution to the assignment problem and its Fair Division solution

Christian Trudeau

Department of Economics, University of Windsor, 401 Sunset Avenue, Windsor, Ontario, Canada

A R T I C L E I N F O

Article history:
Received 20 August 2016
Available online 8 September 2017

JEL classification:
C71
D63

Keywords:
Assignment problems
Bankruptcy problems
Core
Fair division
Concede-and-divide
Minimal rights

A B S T R A C T

We revisit two classic problems: the assignment problem, in which matched pairs of agents create value, and the bankruptcy problem, in which we need to share an endowment among agents with conflicting claims. Since Core Selection constrains us to exactly divide the value created by matched agents, the assignment problem corresponds to multiple two-player bankruptcy problems. From this we obtain equivalence between the Concede-and-Divide (Aumann and Maschler, 1985) sharing method for the bankruptcy problem and the Fair Division solution (Thompson, 1981) for the assignment problem.

In bankruptcy problems, the minimal rights of a claimant is what is left of the endowment when all claimants but himself have received their full claims. By the Minimal Rights First property, it is irrelevant if we distribute the minimal rights first or proceed on the original problem. By adapting the property to the assignment problem, we obtain two characterizations of the Fair Division solution.

1. Introduction

The assignment problem, proposed in Shapley and Shubik (1971), is a transferable-utility version of the matching model of Gale and Shapley (1962). We revisit this classic problem by linking it to a fundamental sharing problem, the bankruptcy problem. While the two problems are well-studied, we provide a clear link between the two. The link allows us to show the concordance of two well-known sharing methods. It also allows for axioms and characterizations for the bankruptcy problem to spill over to the assignment problem.

Assignment problems represent a two-sided matching market, with buyers on one side and sellers on the other. They trade a good in indivisible units. Side payments are allowed and utility is measured in terms of money. Each buyer demands one unit, while sellers each have one unit. Goods are not identical (houses for example) and buyers may have different evaluations of the units up for sale. Sellers may have different reservation prices. The joint profit of matching a buyer with a seller is thus the difference between the evaluation of the buyer and the reserve price of the seller.

A natural way to define a cooperative game from this problem is to suppose that a coalition of buyers and sellers aim to maximize the joint total profits created by these trades. Shapley and Shubik (1971) showed most of the important results used in this paper. They showed, in particular, that the game has a non-empty core and that it includes two special extreme points: a buyer-optimal allocation and a seller-optimal allocation, in which all members of one side simultaneously obtain their most advantageous allocation in the core. In addition, they showed the coincidence of the set of core allocations with

E-mail address: trudeauc@uwindsor.ca.

http://dx.doi.org/10.1016/j.geb.2017.09.005
0899-8256/

the set of competitive equilibria allocations. This implies that in any core allocation, a pair of matched buyer and seller share exactly the surplus they create. There has been further extensive studies of the game (see Roth and Sotomayor (1992) for an overview of the assignment problems, as a special case of two-sided matching problems) and its core (Balinski and Gale, 1987; Hamers et al., 2002; Nunez and Rafels, 2003).

A bankruptcy problem consists in sharing an endowment among agents who have claims for it, with the sum of the claims larger than the endowment. While introduced in the economic literature by O'Neill (1982), the topic has been studied by religious and philosophical scholars for centuries, with many solutions studied by economists having been proposed as far back as in the 14th century. See Thomson (2003, 2015) for reviews of the literature.

For the assignment problem very few allocation methods, which tells us how much money to allocate to each agent for any given problem, have been studied. Among them, van den Brink and Pinter (2015) examine and characterize the Shapley value (Shapley, 1953), which might not be in the core. Nunez and Rafels (2006) provide an allocation method that is always in the core, by first transforming the value matrix to make sure that the Shapley value of the corresponding game is always in the core. Adjustments to individual shares are then made, using the information on how we modified the matrix.

Thompson (1981) defined the Fair Division solution, the average of the buyer-optimal and seller-optimal allocations. Demange (1982) and Leonard (1983) studied these extreme core allocations from a non-cooperative point of view (more precisely their strategyproofness). The solution is further studied in Nunez and Rafels (2002). It is shown that it satisfies two natural monotonicity property (if the value of the match between two agents increases, all else equal, the two agents do not suffer, and if an agent is added to the market, agents on the same side cannot gain while agents on the other side of the market cannot suffer). The solution also coincides with the τ-value, a concept defined for general cooperative games by Tijs (1981), which consists in compromising between an utopia allocation, where all agents get their most favorable allocation compatible with the core, and a minimal-rights vector, where an agent receives what is left after all other members of a coalition have been paid their utopia allocation. This paper provides the first axiomatic characterizations of the Fair Division solution.

For the bankruptcy problem, proposed methods include the Proportional method (endowment shared in proportion to claims), the Constrained Equal Awards method (endowment shared as equally as possible, with no agent receiving more than her claim), the Constrained Equal Losses method (endowment shared such that the losses – claims minus shares – are as equal as possible, with no agent receiving a negative share) and the Random Arrival method (equivalent of the Shapley value (O'Neill, 1982)). Methods combining in some ways the aforementioned methods have also been proposed, among them the Talmud method (Aumann and Maschler, 1985). For the case of two claimants, an interesting method is the Concede-and-Divide solution (Aumann and Maschler, 1985), which consists in first assigning to each agent his minimal right, which consists in what is left, if any, when the other agents receive their full claims. Whatever is left is then divided equally among agents. Both the Random Arrival and Talmud methods coincide with the Concede-and-Divide solution in the case of two claimants.

We link the two problems in the following way. We use Shapley and Shubik's result that in the assignment problem, in any core allocation, if two agents are matched to each other, then their combined share is exactly equal to the value they create together. The problem is thus similar to a two-claimant bankruptcy problem, with the endowment equal to the value they create. As for the claims, we use the maximal core allocations. Our first result is that the prescriptions of the Concede-and-Divide solution coincide with those of the Fair Division solution. A related result is that of Tejada and Àlvarez Mozos (2016), that show that the class of multi-sided Böhm-Bawerk assignment games (in which we match $m \geq 2$ different agents and goods on the same side are identical) is isomorphic to the class of bankruptcy games. They are interested in the method that correspond to the Talmud method for multi-sided Böhm-Bawerk assignment games. While our class of assignment problems is limited to 2 sides, we do not assume that goods are identical. Our approach is also different, as we rewrite assignment problems as multiple 2-agents bankruptcy problems, one for each pair of agents assigned to each other.

A link between the assignment problem and the bargaining problem appears in Rochford (1984), who proposes a model in which agents assigned to each other bargain over the value they jointly create. Each partner makes a threat, and the symmetric Nash bargaining solution is to divide equally the difference between the value created and the sum of their threats. Threats are defined as the highest value created with another partner, net of his/her share. It thus depends on shares of others players, and it is shown that if we start with the buyer-optimal or seller-optimal allocation, and we keep updating the shares according to the bargaining process, it will always converge. The resulting allocation is called a symmetrically pairwise-bargained (SPB) allocation and is both in the core and the kernel. While there may be more than one such allocation, the Fair Division solution is typically not one of them. Rochford (1984) describes the difference between the two approaches as symmetry being either applied to the sides of the market for the Fair Division solution (as it averages the best and worst-case scenarios for both sides) or at the individual level in the bargaining process. But our analysis provides different perspectives. First, the bankruptcy process provides a symmetric treatment of the agents and generates the Fair Division solution. Second, while the process yielding SPB is built on rather optimistic threats (if an agent breaks her assigned match and forms a partnership with another agent, her partner will keep the same share, while she pockets all remaining value), one could also define these threats in a pessimistic manner. At its most pessimistic, an agent believes she would not be able to extract more than her minimal core allocations from other partnership. Then, it is easy to see that the ensuing bargaining process would yield immediately (as threats do not depend on shares anymore) the Fair Division solution. The difference between the two approaches can thus be seen as a difference in how threats are defined.

Next, since the maximal and minimal core allocations for the assignment problems are key to the link with the bankruptcy problems, we provide a new method to obtain these values. We then obtain characterizations of the Fair Division solution that use both the link with the assignment problems, notably through some translation of axioms used in the bankruptcy context, and the method to extract minimal core allocations. Our key axiom is an adaptation of the Minimal Rights First property (Curiel, Maschler and Tijs, 1987). The axiom states that assigning the minimal rights first, adjusting the endowment and sharing what is left should give the same allocations as if we proceed directly from the original problem.

Interestingly, the link between the two problems can also provide a new insight on how market prices can appear in the decentralized version of the assignment problem: if agents know enough about other buyers and sellers and use the principles underlying the Concede-and-Divide and Talmud solutions, found in centuries-old religious texts, they will naturally arrive to equilibrium prices. Given that a 2-player bankruptcy problem is much simpler than an $n \times n$ assignment problem, it is much less of a stretch to suppose that agents can understand the game they are in and naturally converge to this equilibrium.

The paper is divided as follows. In Section 2, we introduce the assignment problem. The bankruptcy problem is defined in Section 3, which also offers a formal link with the assignment problem, and between the Fair Division solution for the assignment problem and the Concede-and-Divide solution for the bankruptcy problem. These links use minimal core allocations, and a new algorithm to calculate those is provided in Section 4. We offer characterizations of the Fair Division solution in Section 5. Various discussions are offered in Section 6, notably the implications of the results on the formation of Walrasian prices for the assignment problem and the similarities between the Fair Division solution and the one proposed by Nunez and Rafels (2006).

2. The assignment problem

We have two sets of agents, N and N'.[1] Without loss of generality, we can assume that $|N| = \left|N'\right|$, with $N = \{1, 2, ..., n\}$ and $N' = \left\{1', 2', ..., n'\right\}$.[2] For each $i \in N$ and $j' \in N'$, we have a value $v_{ij'} \geq 0$ that we obtain if we match these two agents. Let $v = \left(v_{ij'}\right)_{i\in N,\ j'\in N'}$ be the vector of all pairwise values. We slightly abuse language and call v a (value) matrix, as it can be represented as such, with $i \in N$ on the ith row and $j' \in N'$ on the (j')th column.

An assignment problem is $A = (N, N', v)$, the sets of players and the set of matching values. When there is no risk of confusion, we simply identify a problem by v, its value matrix.

We then have an assignment problem to solve: An eligible assignment is a set of pairs a such that if $(i, j') \in a$, there is no k' such that $(i, k') \in a$ or k such that $(k, j') \in a$.

Let $\Omega(N, N')$ be the set of eligible assignments. To find the optimal assignment we need to find a^* such that

$$a^* \in \arg \max_{a\in\Omega(N,N')} \sum_{(i,j')\in a} v_{ij'}.$$

The optimal assignment a^* might not be unique and is obtained through the Hungarian algorithm (Kuhn, 1955; Munkres, 1957). Without loss of generality, we assume that one of the optimal assignments is to assign $i \in N$ to $i' \in N'$, for all $i \in N$. We call $\mathcal{V}$ the set of all such matrices. We exploit this feature, and for $i \in N$, it should be understood that i' is his assigned match.

We use two subsets of matrices defined by Solymosi and Raghavan (2001). We say that a matrix has a dominant diagonal if $v_{ii'} \geq v_{ji'}, v_{ij'}$ for all $j \in N$ and $j' \in N'$. Let $\bar{\mathcal{V}}$ be the set of all such matrices. We say that a matrix has a doubly-dominant diagonal if $v_{ii'} + v_{jk'} \geq v_{ik'} + v_{ji'}$, for all $i, j, k \in N$.

2.1. The associated cooperative game, the core and sharing methods

We wish to share the value created by the grand coalition among the agents. Individual shares should be non-negative and sum up to the value generated by the grand coalition. A solution (or sharing method) assigns shares for every problem A. Formally, a solution is a mapping $y : \mathcal{V} \to \mathbb{R}^{N\cup N'}$ such that $\sum_{i\in N\cup N'} y_i = V(N \cup N', v)$. Again, when there is no risk of confusion, we write $y(v)$ instead of $y(A)$.

For every coalition $R \subseteq N$, $S' \subseteq N'$, define $v^{R,S'}$ as the restriction of v to $R \cup S'$. Define $a^*_{R,S'}$ as (one of) the optimal assignment(s) for the problem $A^{R,S'} = (R, S', v^{R,S'})$. Let $V(R \cup S', v) = \sum_{(i,j')\in a^*_{R,S'}} v_{ij'}$ be the maximum value being created by coalition $R \cup S'$.

[1] When no confusion arises singletons are written without set brackets, e.g. we write $N\backslash i$ instead of $N\backslash \{i\}$.

[2] To model a situation in which one side has more agents than the other, one can just add to the short side of the market agents that have valuations of zero with all potential partners. The resulting operational research problems are equivalent.

However, the sharing problems are not necessarily equivalent, for instance if we use sharing method that depends on the number of agents in the market. Regardless, throughout this work we are interested in core allocations. It is well known that these extra agents with zero valuations all have an allocation of zero in the core. In addition, these extra agents are null players, and their presence (or not) have no impact (as can be seen in subsection 2.1) on the core or on the Fair Division solution studied in this work.

The core of problem $A=(N,N',v)$, labeled as $Core(A)$, is the set of allocations $y \in \mathbb{R}^{N\cup N'}$ such that

$$\sum_{i\in R\cup S'} y_i \geq V(R\cup S',v) \text{ for all } R\subseteq N,\ S'\subseteq N'$$

and $\sum_{i\in N\cup N'} y_i = V(N\cup N',v)$. We define the Core Selection property as follows:

Core Selection: For all problems A, $y(A)\in Core(A)$.

It is known since Shapley and Shubik (1971) that if i is matched with i', then an allocation y that is in the core is such that $y_i + y_{i'} = v_{ii'}$, the value they generate when matched to each other. Let $y_i^{\max}(v)$ and $y_i^{\min}(v)$ be, respectively, the largest and smallest core allocations for agent i. From Demange (1982) and Leonard (1983), we know that

$$y_i^{\max}(v) = V(N\cup N',v) - V(N\backslash i\cup N',v) \text{ for all } i\in N$$
$$y_{i'}^{\max}(v) = V(N\cup N',v) - V(N\cup N'\backslash i',v) \text{ for all } i'\in N'.$$

Combining with $y_i + y_{i'} = v_{ii'}$, we obtain

$$y_i^{\min}(v) = v_{ii'} - V(N\cup N',v) + V(N\cup N'\backslash i',v) \text{ for all } i\in N$$
$$y_{i'}^{\min}(v) = v_{ii'} - V(N\cup N',v) + V(N\backslash i\cup N',v) \text{ for all } i'\in N'.$$

We introduce the buyer-optimal and the seller-optimal allocations, first described by Shapley and Shubik (1971). Let y^N be defined as follows: $y_i^N(v) = y_i^{\max}(v)$ for all $i\in N$ and $y_{i'}^N(v) = y_{i'}^{\min}(v)$ for all $i'\in N'$. Let $y^{N'}(v)$ be defined as follows: $y_{i'}^{N'}(v) = y_{i'}^{\max}(v)$ for all $i'\in N'$ and $y_i^{N'}(v) = y_i^{\min}(v)$ for all $i\in N$. Following Thompson (1981), we define the Fair Division solution as the average of these allocations most favorable to agents on one side of the market. Formally, let $y^{FD}(v) \equiv \frac{y^N(v)+y^{N'}(v)}{2} = \frac{y^{\max}(v)+y^{\min}(v)}{2}$. The solution is proved to be the τ-value (introduced by Tijs, 1981) in Nunez and Rafels (2002).

3. The assignment problem as a bankruptcy problem

3.1. Bankruptcy problems

Bankruptcy problems are well studied, and consist in one of the purest examples of sharing problems, being appropriate to share an estate among siblings or the cost of a public project among a group that can jointly afford it. Formally, let M be a set of agents. Let $c\in\mathbb{R}^M$ be the vector of claims and $E\in\mathbb{R}_+$ be the endowment, with $E\leq\sum_{i\in M} c_i$. A bankruptcy problem is $B=(M,c,E)$.

3.2. Assignment problems as a series of 2-player bankruptcy problems

Consider an assignment problem $A=(N,N',v)$, with $v\in\mathcal{V}$. As previously discussed, Core Selection imposes the constraint that $y_i + y_{i'} = v_{ii'}$ for all $i\in N$. We can thus view an assignment problem as a series of bankruptcy problems between pairs of agents i and i'. As for claims, the obvious candidate is to use the maximal core allocations, as, if we use Core Selection as a focal property, this is the maximum an agent can aspire to. Formally, each assignment problem A can be transformed into a series of bankruptcy problems $B_{ii'}^A$, one for each pair of agents (i,i') and such that $B_{ii'}^A = \left(\left\{i,i'\right\},\left\{y_i^{\max}(v), y_{i'}^{\max}(v)\right\}, v_{ii'}\right)$.

3.3. Solutions for two-player bankruptcy problems

Note that the set of bankruptcy problems that we obtain from assignment problems is a strict subset of bankruptcy problems: not only are we limited to two-player problems, but we also have, by definition, that $c_i\leq E$ for all $i\in M$. Let $\mathcal{B}$ be the set of all such bankruptcy problems. An allocation $\gamma\in\mathbb{R}_+^M$ is such that $\sum_{i\in M}\gamma_i = E$. A sharing method for bankruptcy problems, $\gamma(B)$ assigns an allocation to every bankruptcy problem B. Formally, a sharing method is a mapping $\gamma:\mathcal{B}\to\mathbb{R}_+^M$ such that $\sum_{i\in M}\gamma_i = E$. Among the large set of solutions proposed for the bankruptcy problem, we focus on the two-player solutions and adapt them to our set $\mathcal{B}$.

The paper will mostly focus on the Concede-and-Divide solution (Aumann and Maschler, 1985) which consists in first assigning to each agent the difference between the endowment and the claim of the other player, before dividing the remainder equally.

For all $B\in\mathcal{B}$, $\gamma_i^{CD}(B) = E - c_{i'} + \frac{c_i+c_{i'}-E}{2} = \frac{c_i-c_{i'}+E}{2}$ and similarly $\gamma_{i'}^{CD}(B) = \frac{c_{i'}-c_i+E}{2}$.

Over the set of two-player bankruptcy problems, the Concede-and-Divide solution coincide with many well-studied solutions, notably the Talmud rule (Aumann and Maschler, 1985) and the Random Arrival rule (O'Neill, 1982).

Another popular solution is the Constrained Equal Losses rule, that tries to equalize the loss (claim minus allocation) of each agent. For general two-player problems it is often impossible to equalize the losses, we then prioritize the agent with the larger claim, to bring his loss as close as possible to the loss of the smaller claimant. With our constraint that $c_i \le E$ for all $i \in M$, this is never a problem and we can always achieve equal losses. We thus have that

$$\begin{aligned} c_i - \gamma_i^{CEL} &= c_{i'} - \gamma_{i'}^{CEL} \\ &= c_{i'} - E + \gamma_i^{CEL} \end{aligned}$$

which simplifies to $\gamma_i^{CEL} = \frac{c_i - c_{i'} + E}{2} = \gamma_i^{CD}$. Thus, within $\mathcal{B}$, the Concede-and-Divide and (Constrained) Equal Losses solutions are one and the same.

We complete this subsection by describing two other popular solutions for bankruptcy problems that do not coincide with Concede-and-Divide within $\mathcal{B}$.

The Constrained Equal Awards rule is such that we want to have allocations as equal as possible, under the constraint that no agent receives more than her claim. Thus, $\gamma_i^{CEA} = \min\{c_i, \lambda\}$, with λ chosen so that $\gamma_i^{CEA} + \gamma_{i'}^{CEA} = E$.

The proportional rule divides the endowment in proportion to claims: $\gamma_i^P = \frac{c_i}{c_i + c_{i'}} E$.

3.4. Fair Division is Concede-and-Divide

We conclude this section by showing that the Fair-Division solution for assignment problems is equivalent to the Concede-and-Divide solution for the corresponding bankruptcy problems.

Theorem 1. *For all assignment problem $A = (N, N', v)$, $y_i^{FD}(A) = \gamma_i^{CD}(B_{ii'}^A)$ for all $i \in N$ and $y_{i'}^{FD}(A) = \gamma_{i'}^{CD}(B_{ii'}^A)$ for all $i' \in N'$.*

Proof. Let $A = (N, N', v)$ and $i \in N$. We have that

$$\begin{aligned} \gamma_i^{CD}(B_{ii'}^A) &= \frac{c_i - c_{i'} + E}{2} \\ &= \frac{y_i^{\max}(v) - y_{i'}^{\max}(v) + v_{ii'}}{2} \\ &= \frac{y_i^{\max}(v) + y_i^{\min}(v)}{2} \\ &= \frac{y_i^N(A) + y_i^{N'}(A)}{2} \\ &= y_i^{FD}(A). \end{aligned}$$

We similarly obtain that $y_{i'}^{FD}(A) = \gamma_{i'}^{CD}(B_{ii'}^A)$ for all $i' \in N'$. $\square$

4. Minimal core allocations and dominant diagonal matrices for assignment problems

Minimal core allocations for the assignment problem are crucial for our analysis. In this section, we propose a new algorithm to obtain them.

In Section 2, we provided expressions of the minimal core allocations that depend on values created by the grand coalition and by coalitions that exclude exactly one agent. Thus, to obtain the minimal core allocations for all agents on one side of the market, we need to solve for the optimal assignments in $n+1$ different assignment problems (one that includes all agents, and n problems that each exclude one agent). That is quite demanding computationally.

An alternative is provided by Demange (1982): Interpreting N' as the set of buyers, N as the set of objects and y_i as the price for object i, she starts with all prices at zeros. She then looks at which objects maximize the consumer surplus of each buyer j': $(v_{ij'} - y_i)$. If all agents demand their assigned object, we stop. Otherwise, we find a set of overdemanded objects (such that a set of agents demand only these objects, with the set of agents larger than the set of objects) and increase their prices until one of the agents demands an object outside of the set. We keep iterating on the price vector until every agent demands its assigned object. The price vector that we obtain is exactly $\{y_i^{\min}\}_{i \in N}$.

The algorithm we propose starts with a slightly different objective, but turns out to be a variant of Demange's algorithm. Suppose that we wish to make each element of the diagonal the maximal element on its column. If it's not true, there is an agent $j \in N$ such that $v_{ji'} - v_{ii'} > 0$. We reduce all values created by j by $v_{ji'} - v_{ii'}$ and repeat the process until that objective is attained. We can then use a symmetrical algorithm to make each diagonal element the maximal element on its row. What we obtain is by definition a dominant diagonal matrix. It is worth noting that the objectives of making every diagonal elements the maximal element on its column and of making it the maximal element in its row are independent.[3]

[3] While making changes to make each diagonal element the maximal element in its column, we substract from all elements in a row the same number (with possible censoring at zero). Therefore, we do not alter the rankings within elements of that row.

We obtain the minimal core allocations as a by-product of the algorithm: they are the extra values that prevented a diagonal element to be maximal on its row or column. One can see the link with Demange's algorithm, as making the diagonal element the maximal element on its row is equivalent to making sure that an agent demands the object that he was assigned. The link between minimal core allocations and dominant diagonal matrices will be exploited in the next section to characterize the Fair Division solution. Our formulation also allows to show that the algorithm terminates in at most n steps. This result is formally proven in appendix. The main gain with respect to Demange's algorithm is that instead of looking for an overdemanded set of agents at each step, our computations are strictly at the individual level and thus simpler.

We need the following definitions: Let $D_{i'}(v) = \arg\max_{j \in N} v_{ji'}$ for all $i' \in N'$. In words, $D_{i'}(v)$ is the set of agents that create the largest value with i'. Notice that by definition if $i \in D_{i'}(v)$ for all $i \in N$ then v is such that every diagonal element is maximal in its column.

Algorithm 1. Start with an assignment problem $A = (N, N', v)$.

Step 1: If $i \in D_{i'}(v)$ for all $i' \in N'$, let $e^1 = 0^N$, $\hat{v} = v$ and terminate. Otherwise, let $e_i^1 = \max_{j \in N} v_{ij'} - v_{jj'}$ for all $i \in N$ and v^2 be such that $v_{ij'}^2 = \max\left\{0, v_{ij'} - e_i^1\right\}$ for all $i \in N$ and $j' \in N'$; and proceed to step 2.

Step $k > 1$: If $i \in D_{i'}(v^k)$ for all $i' \in N'$, let $e^k = 0^N$, $\hat{v} = v^k$ and terminate. Otherwise, let $e_i^k = \max_{j \in N} v_{ij'}^k - v_{jj'}^k$ for all $i \in N$, v^{k+1} be such that $v_{ij'}^{k+1} = \max\left\{0, v_{ij'}^k - e_i^k\right\}$ for all $i \in N$ and $j' \in N'$; and proceed to step $k + 1$.

The outputs are the vector $\hat{e}(v) = \sum e^k \in \mathbb{R}_+^N$ and the matrix $\hat{v}$. We then start with the assignment problem $\hat{A} = (N, N', \hat{v})$ and subject it to a symmetrical algorithm designed to make every diagonal element the maximal element on its row. The outputs of that algorithm are the vector $\bar{e}(v) = \sum e^k \in \mathbb{R}_+^{N'}$ and the matrix $\bar{v} \in \bar{\mathcal{V}}$. Let $e(v) \in \mathbb{R}_+^{N \cup N'}$ be such that $e_i(v) = \hat{e}_i(v)$ for all $i \in N$ and $e_{i'}(v) = \bar{e}_{i'}(v)$ for all $i' \in N'$.

It is worth nothing that Leonard (1983) wrote the problem of finding minimal core allocations for one side of the market as a linear program, in which we want to minimize the total share for all agents on that side of the market, subject to the core constraints for each pair of agents. Leonard (1983) does not provide an algorithm to solve the problem. If we use the fact that $y_i + y_{i'} = v_{ii'}$, the core constraints for coalition $\left\{i, j'\right\}$ becomes $y_i - y_j \geq v_{ij'} - v_{jj'}$ for all $i, j \in N$. We can see the link with our algorithm, in which we extract those values $v_{ij'} - v_{jj'}$ as the minimal core allocations.

We illustrate the algorithm with the following example:

Example 1. Let $N = \{1, 2, 3, 4\}$ and $N' = \left\{1', 2', 3', 4'\right\}$ and v be as follows:

$v_{ij'}$	$1'$	$2'$	$3'$	$4'$
1	7	6	1	7
2	2	3	5	1
3	0	2	4	0
4	6	2	7	7

We can easily see that an optimal assignment is on the diagonal. However, the matrix does not have a dominant diagonal. We have that $V(N \cup N', v) = 21$.

We apply the algorithm on (N, N', v).

Step 1: We do not have that $i \in D_{i'}(v)$ for all $i' \in N'$. We have $e_1^1 = v_{12'} - v_{22'} = 3$, $e_2^1 = v_{23'} - v_{33'} = 1$, $e_4^1 = v_{43'} - v_{33'} = 3$ and $e_3^1 = 0$. We have v^2 as follows:

$v_{ij'}^2$	$1'$	$2'$	$3'$	$4'$
1	4	3	0	4
2	1	2	4	0
3	0	2	4	0
4	3	0	4	4

Step 2: We do not have that $i \in D_{i'}(v^2)$ for all $i' \in N'$. We have $e_1^2 = v_{12'}^2 - v_{22'}^2 = 1$ and $e_2^2 = e_3^2 = e_4^2 = 0$. We obtain v^3:

$v_{ij'}^3$	$1'$	$2'$	$3'$	$4'$
1	3	2	0	3
2	1	2	4	0
3	0	2	4	0
4	3	0	4	4

Step 3: Since $i \in D_{i'}(v^3)$ for all $i' \in N'$, we terminate. We have $\hat{v} = v^3$. Thus, $\hat{e}_1 = 3+1 = 4$, $\hat{e}_2 = 1+0 = 1$, $\hat{e}_3 = 0$ and $\hat{e}_4 = 3+0 = 3$.

We now apply the symmetric algorithm to $(N, N', \hat{v})$.

Step 1: We have $e^1_{3'} = \hat{v}_{23'} - \hat{v}_{22'} = 2$ and $e^1_{1'} = e^1_{2'} = e^1_{4'} = 0$. We have $\hat{v}^2$ as follows:

$\hat{v}^2_{ij'}$	$1'$	$2'$	$3'$	$4'$
1	3	2	0	3
2	1	2	2	0
3	0	2	2	0
4	3	0	2	4

Step 2: We have that $i' \in D_i(\hat{v}^2)$ for all $i \in N$, so we terminate. We have $\bar{v} = \hat{v}^2$ and $\bar{e}_{1'} = 0$, $\bar{e}_{2'} = 0$, $\bar{e}_{3'} = 2$ and $\bar{e}_{4'} = 0$. Notice that $\bar{v} \in \bar{\mathcal{V}}$.

It turns out that the vector $e(v)$ is the vector of minimal core allocations. To show this result formally, we need the following lemma, that describes how the core behaves when we subtract from the values a vector that is no larger than the vector of minimal core allocations.

Lemma 1. *For all $a \leq y^{\min}(v)$ and v^{-a} such that $v^{-a}_{ij'} = \max\{0, v_{ij'} - a_i - a_{j'}\}$ for all $i \in N$, $j' \in N'$, we have that $Core(v) = Core(v^{-a}) + a$ and $y^{\min}(v) = y^{\min}(v^{-a}) + a$.*

Proof. Per Shapley and Shubik (1971), for allocations with non-negative shares, the relevant core constraints are those for pairs of agents. We thus have that $x \in Core(v)$ if $x_k + x_{l'} \geq v_{kl'}$ for all $k \in N$, $l' \in N'$.

Nunez and Rafels (2006) show that if $v_{ij'} - a_i - a_{j'} \geq 0$, we have that $x_i + x_{j'} \geq v_{ij'} \Leftrightarrow x_i - a_i + x_{j'} - a_{j'} \geq v^{-a}_{ij'}$ We show that it is also true if $v_{ij'} - a_i - a_{j'} < 0$.

By definition of $y^{\min}$, if $x \in Core(v)$, then $x \geq y^{\min}(v)$. We thus have that $x_i + x_{j'} \geq y^{\min}_i(v) + y^{\min}_{j'}(v)$ and by assumption $y^{\min}_i(v) + y^{\min}_{j'}(v) \geq a_i + a_{j'} > v_{ij'}$. We thus also have that $x_i - a_i + x_{j'} - a_{j'} \geq 0 = v^{-a}_{ij}$ and thus $x - a \in Core(v^{-a})$.

It remains to show that $x - a \in Core(v^{-a}) \Rightarrow x \in Core(v)$. If $x - a \in Core(v^{-a})$, then for all $i \in N$ and $j' \in N'$, $x_i - a_i + x_{j'} - a_{j'} \geq v^{-a}_{ij'} = \max\{0, v_{ij'} - a_i - a_{j'}\}$. This can be written as

$$x_i + x_{j'} \geq \max\{a_i + a_{j'}, v_{ij'}\} \geq v_{ij'}$$

and thus $x \in Core(v)$.

Thus, $x \in Core(v) \Leftrightarrow x - a \in Core(v^{-a})$. This implies that $Core(v) = Core(v^{-a}) + a$ and $y^{\min}(v) = y^{\min}(v^{-a}) + a$. □

We are now ready to show that $e(v)$ obtained in our algorithm is equal to $y^{\min}(v)$.

Theorem 2. *For any $v \in \mathcal{V}$, $e(v) = y^{\min}(v)$.*

Proof. We first expand the definition of e^k to a vector $\tilde{e}^k \in \mathbb{R}^{N \cup N'}_+$ in the following way: $\tilde{e}^k_i = e^k_i$ for all $i \in N$ and $\tilde{e}^k_{i'} = 0$ for all $i' \in N'$.

Suppose that at step k in Algorithm 1, we have $e^k_i = v^k_{ij'} - v^k_{jj'}$. It implies that we have $V(\{j, j'\}, v^k) = v^k_{jj'}$ and $V\{(i, j, j'\}, v^k) = v^k_{ij'}$. The core constraints imposes $y_i + y_j + y_{j'} \geq v^k_{ij'}$ and $y_j + y_{j'} = v^k_{jj'}$. Combining, we obtain $y_i \geq v^k_{ij'} - v^k_{jj'}$. The minimal core allocation for agent i is thus at least as large as $v^k_{ij'} - v^k_{jj'}$. Since the core is non-empty, we cannot have the minimal core allocation to be smaller than this bound. Thus, at each stage k, $\tilde{e}^k \leq y^{\min}(v^k)$.

By Lemma 1, we obtain that at each stage, $Core(v^k) = Core(v^{k+1}) + \tilde{e}^k$ and $y^{\min}(v^k) = y^{\min}(v^{k+1}) + \tilde{e}^k$. Let K be the stage at which the algorithm terminates. At stage K, we have, by definition, that $y^{\min}_i(\hat{v}) = 0$ and $y^{\min}_{i'}(\hat{v}) = y^{\min}_{i'}(v)$ for all $i \in N$. Thus $y^{\min}(v^{K-1}) = \tilde{e}^{K-1}$. We then obtain $y^{\min}(v^{K-2}) = y^{\min}(v^{K-1}) + \tilde{e}^{K-2} = \tilde{e}^{K-1} + \tilde{e}^{K-2}$. In the same manner, we obtain, for all $k = 1, ..., K-1$ that

$$y^{\min}(v^{K-k}) = \sum_{l=1}^{k} \hat{e}^{K-l}.$$

In particular, for $k = K-1$, we obtain $y^{\min}(v) = \sum_{k=1}^{K-1} \tilde{e}^{K-k}$. For all $i \in N$, we obtain that $y^{\min}_i(v) = \sum_{k=1}^{K-1} e^{K-k}_i = e_i(v)$.

Applying the same procedure to the symmetric version of Algorithm 1, supposing that it terminates in K' stages, we obtain that for all $i' \in N'$, $y^{\min}_{i'}(v) = \sum_{k=1}^{K'-1} e^{K'-k}_{i'} = e_{i'}(v)$. □

5. Characterizations of the Fair Division solution

We now introduce a series of properties that will be used to characterize the Fair Division solution. We start with a simple symmetry property. Suppose that agents $i \in N$ and $i' \in N'$ are symmetric in the following sense: the value created by i and agent k' is exactly the same as the one obtained by i' and k, for all $k \in N$. Then, given that i and i' have the same possible deviations, they should receive the same allocation.

Symmetry. For all $A = (N, N', v)$, if v is such that $v_{ki'} = v_{ik'}$ for all $k \in N$, then $y_i(v) = y_{i'}(v)$.

It is natural to have shares depend on the outside options, which in our case are the matching values with partners other than the assigned partner. In particular, an agent should be assigned a larger share if many agents on the other side want to be matched with him, because he creates high values with many partners. In those cases, the threat that the agent can defect from the optimal assignment is credible. However, in the case in which a pair of matched agents cannot create larger values with other partners, than the threat of defection is empty. Suppose that the value matrix changes and that we have that before and after the change i) a pair of agents is matched together and create the same value and ii) their threats to deviate with other partners are empty. We argue that when these two conditions are met, for all the pairs of matched agents, the situations are identical and the allocations should remain the same as the value matrix changes.

Independence of Empty Threats. If v and $\tilde{v}$ are such that $v_{ii'} = \tilde{v}_{ii'} \geq v_{ij'}, v_{ji'}, \tilde{v}_{ij'}, \tilde{v}_{ji'}$ for all $i, j \in N$, then $y(v) = y(\tilde{v})$.

In particular, Independence of Empty Threats states that if two matrices with dominant diagonals have identical values on the diagonal, then the shares should be the same, as elements outside the diagonal (the so-called empty threats) are irrelevant.

Our final two properties take advantage of the link between assignment and bankruptcy problems, as discussed in Section 3. The first is based on the Minimal Rights First property for bankruptcy problems (Curiel, Maschler, Tijs, 1987). The Minimal Rights First property considers the case in which everybody but agent i has received their full claims. If the resource has not been exhausted, then agent i should clearly receive what is left, agent i's so-called minimal right. The property then says that we should obtain the same shares if we i) apply our solution to the original problem or ii) assign everybody their minimal right, reduce their claims and the endowment accordingly and apply our solution to the new problem. The equivalent concept for assignment problems is for shares to be invariant to our treatment of the minimal core allocations: we obtain the same shares if we i) apply our solution to the original problem or ii) subtract from all values the minimal core allocations (with censorship to zero if needed) and apply our solution to the new assignment problem.

Minimal Core Allocations First. For any $v \in \mathcal{V}$, $y(v) = y(\bar{v}) + y^{\min}(v)$.

Our final property adapts the Reasonable Lower Bound property of Moreno-Ternero and Villar (2004) and Yeh (2008). The justification in both the bankruptcy and the assignment problems is that in order to ensure that agents will accept to voluntarily participate, offering a guarantee helps tremendously. While some agents cannot receive a strictly positive share (those for which the maximal core allocation is zero), we can guarantee a strictly positive share to others, even if their minimal core allocation is zero. In particular, we can guarantee half of the maximal core allocations. Since the property is translated from the bankruptcy problem property of Moreno-Ternero and Villar (2004) and Yeh (2008), for which it provides the largest possible guarantee, we obtain the same result for allocation problems. In particular, we might have that both partners claim the full value they create together, which implies that we cannot guarantee more than half of the maximal core allocations.

Reasonable Lower Bound. For any $v \in \mathcal{V}$ we have $y(v) \geq \frac{y^{\max}(v)}{2}$.

Using the fact that $y_i^{\max}(v) = V(N \cup N', v) - V(N \backslash i \cup N', v)$, we can also reinterpret the property as $y_i(v) \geq \frac{V(N \cup N', v) - V(N \backslash i \cup N', v)}{2}$ for all $i \in N$. In words, each agent should receive at least half of the extra value she creates when she joins the grand coalition. One can then interpret the property as follows: i is obviously at least partly responsible for that incremental value, but she cannot create that value by herself, as she needs $N \backslash i \cup N'$. The property allows for partial responsibility, but clearly defines the lower bound at half of that incremental value.

We are now ready for our first characterization.

Theorem 3. *A solution y satisfies Symmetry, Core Selection, Independence of Empty Threats and Minimal Core Allocations First if and only if $y = y^{FD}$.*

Proof. We first show that y^{FD} satisfies the properties.

Symmetry: Suppose that v is such that $v_{ki'} = v_{ik'}$ for all $k \in N$. Then, $y_i^{\min}(v) = y_{i'}^{\min}(v)$ and $y_i^{\max}(v) = y_{i'}^{\max}(v)$, yielding that $y_i^{FD}(v) = y_{i'}^{FD}(v)$.

Core Selection: By Lemma 1, we only need to show that it is satisfied for $v \in \bar{\mathcal{V}}$. Following Shapley and Shubik (1971), we only need to verify coalitions consisting of one player on each side of the market. Take $i \in N$ and $j' \in N'$. $y_i^{FD}(v) + y_{j'}^{FD}(v) = \frac{v_{ii'}}{2} + \frac{v_{jj'}}{2} \geq v_{ij'}$, since, by definition of $\bar{\mathcal{V}}$, $v_{ii'}, v_{jj'} \geq v_{ij'}$.

Independence of Empty Threats: Let v and $\tilde{v}$ be such that $v_{ii'} = \tilde{v}_{ii'} \geq v_{ij'}, v_{ji'}, \tilde{v}_{ij'}, \tilde{v}_{ji'}$ for all $i, j \in N$. Then, $y_i^{FD}(v) = \frac{v_{ii'}}{2} = \frac{\tilde{v}_{ii'}}{2} = y_i^{FD}(\tilde{v})$ for all $i \in N$. We obtain the same result for $i' \in N'$.

Minimal Core Allocations First: For all $i \in N$ we have:

$$\begin{aligned} y_i^{FD}(v) &= \frac{y_i^{\max}(v) + y_i^{\min}(v)}{2} \\ &= \frac{y_i^{\max}(\bar{v}) + y_i^{\min}(\bar{v})}{2} + y_i^{\min}(v) \\ &= y_i^{FD}(\bar{v}) + y_i^{\min}(v), \end{aligned}$$

where the second equality comes from Lemma 1. We obtain the same result for $i' \in N'$.

We show that a unique solution satisfies the properties. Suppose that y satisfies all properties.

By Minimal Core Allocations First, $y(v) = y(\bar{v}) + y^{\min}(v)$. By Lemma 1 and Theorem 2, $\bar{v} \in \bar{\mathcal{V}}$. Let $\bar{v}^0$ be such that $\bar{v}^0_{ii'} = \bar{v}_{ii'}$ for all $i \in N$ and $\bar{v}^0_{ij'} = 0$ if $i \neq j$. By Independence of Empty Threats, we have that $y(\bar{v}) = y(\bar{v}^0)$. By Core Selection, we have that $y_i(\bar{v}^0) + y_{i'}(\bar{v}^0) = \bar{v}^0_{ii'} = \bar{v}_{ii'}$ for $i = 1, ..., n$. By Symmetry, we must have that $y_i(\bar{v}^0) = y_{i'}(\bar{v}^0)$. Combining these two results, we obtain $y_i(\bar{v}^0) = y_{i'}(\bar{v}^0) = \frac{\bar{v}_{ii'}}{2}$ for $i = 1, ..., n$. Thus, we have a unique solution defined for all $v \in \mathcal{V}$. □

Notice that we barely use Core Selection in the proof. We can replace it by the weaker property of Submarket Efficiency (van den Brink and Pinter, 2015), which says that if we can divide $N \cup N'$ in subgroups such that sellers in one subgroup have no gains to match to buyers in other groups, and vice-versa, then the members of each subgroup will split the value they jointly generate. Since we focus on the core in our analysis, we use the stronger Core Selection property.

We can also forego Core Selection, as well as Independence of Empty Threats, if we are ready to strengthen Symmetry. We replace it with Reasonable Lower Bound, which, in opposition to Symmetry, applies to all matrices and suggests some form of symmetry. This second characterization closely follows the characterization of the Concede-and-Divide solution for bankruptcy problems obtained by Yeh (2008).

Theorem 4. *A solution y satisfies Reasonable Lower Bound and Minimal Core Allocations First if and only if $y = y^{FD}$.*

Proof. We have shown in Theorem 3 that y^{FD} satisfies Minimal Core Allocations First. We show that it satisfies Reasonable Lower Bound. We have that $y^{FD} = \frac{y^{\max}(v)+y^{\min}(v)}{2} \geq \frac{y^{\max}(v)}{2}$ since $y^{\min}(v) \geq 0^{N \cup N'}$.

We show that a unique solution satisfies the properties. Suppose that y satisfies Reasonable Lower Bound and Minimal Core Allocations First. By Minimal Core Allocations First, $y(v) = y(\bar{v}) + y^{\min}(v)$. By definition, $y^{\min}(\bar{v}) = 0^{N \cup N'}$ and thus, for all $i \in N$, $y_i^{\max}(\bar{v}) = y_{i'}^{\max}(\bar{v}) = \bar{v}_{ii'}$. For all $i \in N$ we have $y_i(\bar{v}), y_{i'}(\bar{v}) \geq \frac{\bar{v}_{ii'}}{2}$, by Reasonable Lower Bound, and $\sum_{i \in N} y_i(\bar{v}) + \sum_{i' \in N'} y_{i'}(\bar{v}) = \sum_{i \in N} \bar{v}_{ii'}$, by budget balance. The only way to reconcile these two constraints is to have $y_i(\bar{v}) = y_{i'}(\bar{v}) = \frac{\bar{v}_{ii'}}{2}$ for all $i \in N$. Thus, we have a unique solution defined for all $v \in \mathcal{V}$. □

One could obtain the characterization, with the same proof, using a slightly different property proposed by Moreno-Ternero and Villar (2006): if an agent claims the full endowment, he should receive at least half of the endowment. Translated to our problem, it means that if an agent's maximal core allocation is equal to the full value created with his match, he should receive at least half of that value.

The properties of Theorem 3 and of Theorem 4 are independent. Formal proof is in the Appendix A.2.

If we remove Symmetry from Theorem 3, we obtain many other allocations, including the buyer-optimal and seller-optimal allocations, which jointly maximize the welfare of one side of the market. Symmetry implies that both sides of the market are treated in the same manner, which might not always be true. Note however that if we amend the property to recognize that one side of the market has a systematic advantage, it still would not be enough to characterize the full core, which also includes allocations in which one side of the market is favored in one pair, but disfavored in another.

Using Lemma 1, we can show that in both of the theorems above we can replace Minimal Core Allocations First by a property that says that if at least one agent j' creates more value with agent i then with her assigned match, then if we increase all values created by agent i by $\beta > 0$, all else equal, then the share of agent i should increase by exactly β. The property can be interpreted as follows: an agent that is "overdemanded" is in a very desirable position, bargainingwise. Any further increase in the value she creates will go straight to her pockets. Given that we have focused on the core in our analysis, we have kept the weaker Minimal Core Allocations First property. But using this new property instead would allow to obtain Theorem 3 by using properties defined only on v and not on V.

Finally, we remark that for any $\bar{v} \in \bar{\mathcal{V}}$, either the combination of Symmetry, Core Selection, Independence of Empty Threats or Reasonable Lower Bound by itself implies that $y_i(\bar{v}) = y_{i'}(\bar{v}) = \frac{\bar{v}_{ii'}}{2}$. But for $v \in \mathcal{V}$, typically Symmetry and Independence of Empty Threat have no bite, leaving us with the core constraints, which are neither stronger or weaker than the constraints of Reasonable Lower Bound.

6. Discussion

6.1. Fair Division and Walrasian prices[4]

In this subsection, we provide an argument that the Fair Division solution and its link with the Concede-and-Divide solution can be seen as a way for agents to naturally converge towards Walrasian prices. It is still unclear how Walrasian prices form. Walras' suggestion involved the so-called Walrasian auctioneer, that asks buyers and sellers for their quantities demanded and supplied at a given set of prices, and that updates these prices until he finds the equilibrium. Only at that point do trades occur. Research in this area have mostly found negative results, including the possibility that the auctioneer can never converge to the Walrasian prices (Sonnenschein, 1972; Mantel, 1974; Debreu, 1974, to name a few).

While it is known that economic agents tend to choose fair allocations, what constitutes a fair allocation in such a complex model is not clear. This is where our result is useful: if agents realize that the complex assignment problem collapses to a series of 2-player bankruptcy problems, then it's much more realistic to think that agents can converge to equilibrium prices. The information from other agents is condensed in the maximal core allocations, which establishes the bargaining powers within each buyer-seller pair. It is then very natural to assume that we will converge to the prices generating the Concede-and-Divide allocations, as such fair allocations have been shown to be focal points. In addition, the idea itself of Concede-and-Divide has been present in philosophical and religious texts for centuries.

The key difficulty is for agents to determine their maximal core allocations. It is relatively simple if all goods are identical (they then only depend on the value put on the good by the "marginal" buyers and sellers, i.e. those that are almost indifferent between making a transaction or not, see Shapley and Shubik (1971)), it is less obvious in the general problem. The assumptions needed are demanding, but so are those needed in the traditional explanations of the appearance of Walrasian prices.

6.2. Matrices with a dominant and doubly-dominant diagonal

It is worth noting the Fair Division solution has some similarity with the one proposed by Nunez and Rafels (2006). Their idea is to transform a matrix v into one with a matrix that generates an exact game, that is one for which each coalition can obtain exactly its value in a core allocation. In turns, a matrix generates an exact game if and only if it has a dominant diagonal and a doubly-dominant diagonal, with the two properties being independent (Solymosi and Raghavan, 2001). Then, using the fact that the Shapley value of an exact game is in the core (Hoffmann and Sudhölter, 2007) they obtain a core allocation.

They proceed by first transforming v into v^{ddd}, the only matrix with a doubly-dominant diagonal that generates the same core as v. They obtain v^{ddd} as follows. For all $i \in N$ and $j' \in N'$,

$$v^{ddd}_{ij'} = v_{ii'} + v_{jj'} + V(N\backslash j \cup N'\backslash i', v) - V(N \cup N', v).$$

They then further transform the matrix into a matrix with a dominant and doubly-dominant diagonal, v^e, by subtracting the minimal core allocations: For all $i \in N$ and $j' \in N'$, $v^e_{ij'} = v^{ddd}_{ij'} - y^{\min}_i(v) - y^{\min}_{j'}(v)$. Notice that since in moving from v to v^{ddd} we have increased values, there is no need anymore to censor $v^e_{ij'}$ to zero, as $v^{ddd}_{ij'} \geq y^{\min}_i(v) + y^{\min}_{j'}(v)$. Nunez and Rafels thus propose the allocation $y^{NR}(v) = y^{\min}(v) + Sh(v^e)$. For our purposes, since we do not use the Shapley value, the first part of the operation, in which we transform a matrix into one with a doubly-dominant diagonal, is unnecessary.

Consider the following interpretation of the methods, that shows how close the two methods are. It turns out that to obtain the matrix with a dominant and a doubly-dominant diagonal needed for y^{NR}, we can also reverse the order of the operations, first removing the minimal core allocations (to obtain a dominant diagonal) before making changes to obtain a doubly-dominant diagonal. Therefore, we can view both methods as first transforming the matrix into one with a dominant diagonal by extracting the minimal core allocations. To obtain y^{NR}, we then transform the matrix into one with a doubly-dominant diagonal, by making changes to off-diagonal elements, before taking the Shapley value of the resulting game. To obtain the Fair Division solution we transform the off-diagonal elements by making them all equal to zero. Computing the Shapley value of the corresponding cooperative game is equivalent to sharing $v_{ii'}$ equally among the two partners. The difference between the two methods can thus be seen as the transformation made to the matrix with a dominant diagonal before applying the Shapley value.

We contrast our method with the translated Shapley value of Nunez and Rafels (2006) in the following example.

Example 2. We consider a simpler version of Example 1, in which we remove agents 4 and 4′. We have

$$\begin{array}{c|ccc} v_{ij'} & 1' & 2' & 3' \\ \hline 1 & 7 & 6 & 1 \\ 2 & 2 & 3 & 5 \\ 3 & 0 & 2 & 4 \end{array} \quad \text{and} \quad \begin{array}{c|ccc} \bar{v}_{ij'} & 1' & 2' & 3' \\ \hline 1 & 3 & 2 & 0 \\ 2 & 1 & 2 & 2 \\ 3 & 0 & 2 & 2 \end{array}, \quad \text{with } y^{\min}_1 = 4,\ y^{\min}_2 = 1,\ y^{\min}_{3'} = 2,\ y^{\min}_3 = y^{\min}_{1'} = y^{\min}_{2'} = 0.$$

[4] I am indebted to Andrew Mackenzie for this subsection.

We obtain v^e as follows:

$\bar{v}^e_{ij'}$	$1'$	$2'$	$3'$
1	3	2	2
2	1	2	2
3	1	2	2

.

We thus obtain the following shares:

	$y^{FD}(\bar{v})$	$y^{NR}(v^e)$	$y^{FD}(v)$	$y^{NR}(v)$
1	1.5	$1.61\bar{6}$	5.5	$5.61\bar{6}$
2	1	$0.9\bar{3}$	2	$1.9\bar{3}$
3	1	$0.9\bar{3}$	1	$0.9\bar{3}$
$1'$	1.5	$1.38\bar{3}$	1.5	$1.38\bar{3}$
$2'$	1	$1.0\bar{6}$	1	$1.0\bar{6}$
$3'$	1	$1.0\bar{6}$	3	$3.0\bar{6}$

Independence of Empty Threats, used to characterize the Fair Division solution, says that when agents have nothing to gain from deviations, these threats can be seen as empty. It is thus natural to consider the values outside of the optimal assignment as irrelevant for the shares of the agents. Notice that this is in direct opposition with the usual interpretation of the Shapley value, for which all threats are considered as non-empty and relevant, even if they generate much less value. But the core of the assignment problem has two distinctive properties: a pair of matched agents share among themselves the value they generate, and as long as all shares are non-negative, we only need to worry about deviations from pairs. It is because of these characteristics that we can pinpoint much more precisely the threats of deviations than in general cooperative games. This is illustrated clearly by the link with the 2-player bankruptcy problem, in which, by definition, there is nobody to deviate with. Therefore, we believe that our approach is a relevant alternative to allocations based on the Shapley value, such as the method proposed by Nunez and Rafels (2006).

6.3. Other potential links with the bankruptcy problem

While the Concede-and-Divide solution has a nice interpretation as a solution to the assignment problems, what about the other popular solutions for two-player bankruptcy problems? It turns out to be very difficult to satisfy Core Selection, as seen in the following example.

Example 3. We consider again the problem of Example 2. We find the following shares for the Constrained Equal Awards and Proportional solutions:

	γ^{CEA}	γ^{P}
1	4	4.9
2	1.5	1.8
3	2	$1.\bar{3}$
$1'$	3	2.1
$2'$	1.5	1.2
$3'$	2	$2.\bar{6}$

See that $v_{23'} = 5$, but that $\gamma_2^{CEA} + \gamma_{3'}^{CEA} = 3.5$ and $\gamma_2^P + \gamma_{3'}^P = 4.4\bar{6}$. Thus, they do not satisfy Core Selection.

This is particularly problematic, as the link between assignment and bankruptcy problems was obtained with Core Selection at the heart of the argument. This is also not quite surprising, as when we move from a $n \times n$ assignment problem to a 2-player bankruptcy problem, we ignore most of the information of the original problem. That the Concede-and-Divide solution is able to generate an allocation in the core regardless of that is noteworthy. There does not seem to be any other solution for bankruptcy problems that can do the same.

There are many other axiomatizations of the Concede-and-Divide solution for 2-player bankruptcy problems, notably offered by Moreno-Ternero and Villar (2004, 2006). Most of the other properties used are difficult to adapt to our context. Many are consistency properties that apply when the endowment changes, all else equal, or when a claim changes, all else equal. This is difficult to use in our context since claims and endowment usually move together. Another difficulty is that many characterizations use self-duality, a property that makes it such that it does not matter if we allocate the endowment or the losses with respect to the claims. In our case, we lose the property that claims are no larger than the (adjusted) endowment in the dual problem.

6.4. Extensions

The main idea of the paper, that the assignment game can be divided in many 2-player bargaining games, can be extended: as soon as we have a game for which a subgroup of agents obtains a fixed core allocation, the game within that subgroup can be seen as a bankruptcy game, with maximal core allocations serving as claims. In particular, any market game for which there is coincidence between the core and the set of Walrasian allocations can be rewritten as a series of 2-player bankruptcy games.

For instance, while the core might be empty for m-sided assignment problems, if the game is locally additive, i.e. the m sectors are ordered and the worth of a coalition is simply the sum of values created by pairs of agents in consecutive sectors, then in any optimal assignment, any pair of agents in consecutive sectors will exactly share the value they create together. (Stuart, 1997; Atay and Nunez, 2017).

Acknowledgments

The author thanks guest editor David Levine and two anonymous reviewers for their comments. Special thanks to Marina Nunez for precious comments. The author also benefited from comments from Andrew Mackenzie, Tamas Solymosi, seminar participants at University of Barcelona, University of Vigo and University of Guelph, and conference participants at the 5th World Congress of Game Theory. A previous version circulated under the title "Minimal core allocations, empty threats and the Fair Division solution for the assignment problem". This work was supported by the Social Sciences and Humanities Research Council of Canada [grant number 435-2014-0140].

Appendix A

A.1. Efficiency of the algorithm

Theorem 5. *Algorithm 1 concludes in at most n steps.*

Proof. By definition, we have that $v_{ii'}^{k+1} = \max\left\{v_{ii'}^k - e_i^k, 0\right\}$. We first show that $v_{ii'}^k - e_i^k \geq 0$ for all $i \in N$. If $e_i^k = 0$, the result is immediate. Otherwise, there exists $j \in N \backslash i$ such that $e_i^k = v_{ij'}^k - v_{jj'}^k > 0$. We then have that $v_{ii'}^{k+1} = v_{ii'}^k + v_{jj'}^k - v_{ij'}^k$. But, since $v^k \in \mathcal{V}$, an optimal allocation is on the diagonal, which implies that

$$\begin{aligned} v_{ii'}^k + v_{jj'}^k &\geq v_{ij'}^k + v_{ji'}^k \\ &\geq v_{ij'}^k \end{aligned}$$

and thus $v_{ii'}^k - e_i^k \geq 0$, and $v_{ii'}^{k+1} = \max\left\{v_{ii'}^k - e_i^k, 0\right\} = v_{ii'}^k - e_i^k$.

We are now ready to prove the result. We have that $e_i^k = \max_{j \in N} v_{ij'}^k - v_{jj'}^k$.

In particular, this implies that $e_i^1 = \max_{j \in N} v_{ij'} - v_{jj'}$ and

$$\begin{aligned} e_i^2 &= \max_{j \in N} v_{ij'}^2 - v_{jj'}^2 \\ &= \max_{j \in N} \left\{ \max\left\{v_{ij'} - e_i^1, 0\right\} - \max\left\{v_{jj'} - e_j^1, 0\right\} \right\} \\ &= \max_{j \in N} \left\{ \max\left\{v_{ij'} - e_i^1, 0\right\} - v_{jj'} + e_j^1 \right\} \\ &= \max_{j \in N} \left\{ v_{ij'} - e_i^1 - v_{jj'} + e_j^1 \right\} \\ &= \max_{j \in N} \left\{ v_{ij'} - v_{jj'} + e_j^1 \right\} - e_i^1 \\ &= \max_{j \in N} \left\{ v_{ij'} - v_{jj'} + \max_{l \in N} \left\{ v_{jl'} - v_{ll'} \right\} \right\} - e_i^1 \\ &= \max_{j_1, j_2 \in N} \left\{ v_{ij_1'} - v_{j_1 j_1'} + v_{j_1 j_2'} - v_{j_2 j_2'} \right\} - e_i^1 \end{aligned}$$

where the third equality is true since $\max\left\{v_{jj'} - e_j^1, 0\right\} = v_{jj'} - e_j^1$ and the fourth equality is true as if $\max\left\{v_{ij'} - e_i^1, 0\right\} = 0$, then $\max\left\{v_{ij'} - e_i^1, 0\right\} - v_{jj'} + e_j^1 \leq 0 = v_{ii'} - e_i^1 - v_{ii'} + e_i^1$.

Generalizing, we obtain, for all $m \in \{1, 2, ...\}$

$$\sum_{k=1}^{m} e_i^k = \max_{j_1,\dots,j_m \in N} \left\{ v_{ij'_1} - v_{j_1 j'_1} + \sum_{p=2}^{m} v_{j_{p-1} j'_p} - v_{j_p j'_p} \right\}$$

$$= \max \left\{ \sum_{k=1}^{m-1} e_i^k, \max_{\substack{j_1,\dots,j_m \in N\setminus i \\ j_p \neq j_q \text{ for all } p\neq q}} \left\{ v_{ij'_1} - v_{j_1 j'_1} + \sum_{p=2}^{m} v_{j_{p-1} j'_p} - v_{j_p j'_p} \right\} \right\}$$

where the last equality comes from the following facts. If $j_p = j_q$ or if $j_1 = i$, terms cancel out, leading the sum to be equivalent to chains of $m-1$ agents, already covered in the expression of $\sum_{k=1}^{m-1} e_i^k$. If $j_p = i$ for some $p > 1$, say w.l.o.g. $p = m$, we obtain that i is assigned to j'_1, j_{p-1} is assigned to j'_p for all $p = 2, ..., m-1$ and j_{m-1} is assigned to i'. Suppose that all agents in $N\setminus\{j_1, ..., j_m\}$ are assigned to their counterpart on the diagonal, and observe that such an assignment is an eligible assignment for the problem (N, N', v), generating value

$$v_{ij'_1} + \sum_{p=2}^{m-1} v_{j_{p-1} j'_p} + v_{j_{m-1} i'} + \sum_{j \in N\setminus\{j_1,\dots,j_m\}} v_{jj'}.$$

But by definition, an optimal assignment is on the diagonal and generates value of $\sum_{j\in N} v_{jj'}$. Thus, we obtain that

$$v_{ij'_1} + \sum_{p=2}^{m-1} v_{j_{p-1} j'_p} + v_{j_{m-1} i'} + \sum_{j \in N\setminus\{j_1,\dots,j_m\}} v_{jj'} \leq \sum_{j\in N} v_{jj'}$$

$$v_{ij'_1} + \sum_{p=2}^{m-1} v_{j_{p-1} j'_p} + v_{j_{m-1} i'} \leq \sum_{p=1}^{m} v_{j_p j'_p}$$

$$v_{ij'_1} - v_{j_1 j'_1} + \sum_{p=2}^{m} v_{j_{p-1} j'_p} - v_{j_p j'_p} \leq 0.$$

Notice that we then have that $\sum_{k=1}^{n} e_i^k = \sum_{k=1}^{n-1} e_i^k$, as there are only $n-1$ agents in $N\setminus i$, which makes it impossible to pick n agents distinct from each other. Thus, $e_i^n = 0$ for all $i \in N$. Stated otherwise, $v_{ij'}^n \leq v_{jj'}^n$ for all $i, j \in N$, and thus $j \in D_{j'}(v^n)$ for all $j' \in N'$, which is our condition for the algorithm to terminate. □

A.2. Independence of the properties

Lemma 2. *The properties of Theorem 3 are independent.*

Proof. Let y^1 be defined as follows. If $v \in \bar{\mathcal{V}}$ or if $v_{ki'} = v_{ik'}$ for $k = 1, .., n$, then $y_i^1(v) = y_{i'}^1(v) = \frac{v_{ii'}}{2}$ for $i = 1, ..., n$. Otherwise, $y^1(v) = y^{NR}(v)$.

We can verify that y^1 satisfies Symmetry, Core Selection and Independence of Empty Threats but fails Minimal Core Allocations First.

We can verify that y^{NR} satisfies Symmetry, Core Selection and Minimal Core Allocations First but fails Independence of Empty Threats.

Let y^2 be defined as follows. If $v \in \bar{\mathcal{V}}$, then $y_i^2(v) = y_{i'}^2(v) = \frac{V(N\cup N', v)}{2n}$ for $i = 1, ..., n$. If $v \in \mathcal{V}\setminus\bar{\mathcal{V}}$, let $y^2(v) = y^2(\bar{v}) + y^{\min}(v)$.

We can verify that y^2 satisfies Symmetry, Independence of Empty Threats and Minimal Core Allocations First but fails Core Selection.

We can verify that y^N (and $y^{N'}$) satisfies Core Selection, Independence of Empty Threats and Minimal Core Allocations First but fails Symmetry. □

The following table summarizes the lemma: ("+" indicates that the property is satisfied and "−" that it is not)

	SYM	CS	IET	MCAF
y^{FD}	+	+	+	+
y^1	+	+	+	−
y^{NR}	+	+	−	+
y^2	+	−	+	+
y^N	−	+	+	+
$y^{N'}$	−	+	+	+

Lemma 3. *The properties of Theorem 4 are independent.*

Proof. We can verify that y^{NR} satisfies Minimal Core Allocations First but fails Reasonable Lower Bound.

Let y^3 be defined as follows: $y^3_i(v) = y^3_{i'} = \frac{v_{ii'}}{2}$ for all $v \in \mathcal{V}$ and all $i \in N$. We can verify that y^3 satisfies Reasonable Lower Bound but fails Minimal Core Allocations First. □

References

Atay, A., Nunez, M., 2017. Multi-Sided Assignment Games on m-Partite Graphs. University of Barcelona. Economics Working Papers 2017/357.
Aumann, R., Maschler, M., 1985. Game theoretic analysis of a bankruptcy problem from the Talmud. J. Econ. Theory 36, 195–213.
Balinski, M.L., Gale, D., 1987. On the core of the assignment game. In: Functional Analysis, Optimization and Mathematical Economics. Oxford University Press, New York, pp. 274–289.
Curiel, I.J., Maschler, M., Tijs, S.H., 1987. Bankruptcy games. Z. Oper.-Res. 31, 143–159.
Debreu, G., 1974. Excess demand functions. J. Math. Econ. 1, 15–21.
Demange, G., 1982. Strategyproofness in the assignment market game. Laboratoire d'Économie de l'École Polytechnique, Paris. Mimeo.
Gale, D., Shapley, L., 1962. College admissions and the stability of marriage. Am. Math. Mon. 69, 9–15.
Hamers, H., Klijn, F., Solymosi, T., Tijs, S., Villar, J.P., 2002. Assignment games satisfy the CoMa-property. Games Econ. Behav. 38, 231–239.
Hoffmann, M., Sudhölter, P., 2007. The Shapley value of exact assignment games. Int. J. Game Theory 35, 557–568.
Kuhn, H., 1955. The Hungarian method for the assignment problem. Nav. Res. Logist. Q. 2, 83–97.
Leonard, H.B., 1983. Elicitation of honest preferences for the assignment of individuals to positions. J. Polit. Economy 91, 461–479.
Mantel, R., 1974. On the characterization of aggregate excess-demand. J. Econ. Theory 7, 348–353.
Moreno-Ternero, J., Villar, A., 2004. The Talmud rule and the securement of agents' awards. Math. Soc. Sci. 47, 245–257.
Moreno-Ternero, J., Villar, A., 2006. New characterizations for a classical bankruptcy rule. Rev. Econ. Design 10, 73–84.
Munkres, J., 1957. Algorithms for the assignment and transportation problems. J. Soc. Ind. Appl. Math. 5, 32–38.
Nunez, M., Rafels, C., 2002. The assignment game: the τ-value. Int. J. Game Theory 31, 411–422.
Nunez, M., Rafels, C., 2003. Characterization of the extreme core allocations of the assignment game. Games Econ. Behav. 44, 311–331.
Nunez, M., Rafels, C., 2006. A Canonical Representation for the Assignment Game: the Kernel and the Nucleolus. Barcelona Economics Working Paper 279. 23 pp.
O'Neill, B., 1982. A problem of rights arbitration from the talmud. Math. Soc. Sci. 2, 345–371.
Rochford, S., 1984. Symmetrically pairwise-bargained allocations in an assignment market. J. Econ. Theory 34, 262–281.
Roth, A.E., Sotomayor, M.A.O., 1992. Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis. Econom. Soc. Monogr. Cambridge University Press. 280 pp.
Shapley, L.S., 1953. A value for n-person games. In: Kuhn, H.W., Tucker, A.W. (Eds.), Contributions to the Theory of Games. Volume 2. Princeton University Press.
Shapley, L.S., Shubik, M., 1971. The assignment game I: the core. Int. J. Game Theory 1, 9–25.
Solymosi, T., Raghavan, T.E.S., 2001. Assignment games with stable cores. Int. J. Game Theory 30, 177–185.
Sonnenschein, H., 1972. Market excess demand functions. Econometrica 40, 549–563.
Stuart, H., 1997. The supplier-firm-buyer game and its m-sided generalization. Math. Soc. Sci. 34, 21–27.
Tejada, O., Àlvarez-Mozos, M., 2016. Vertical syndication-proof competitive prices in multilateral assignment markets. Rev. Econ. Design 20, 289–327.
Thompson, G.L., 1981. Auctions and market games. In: Aumann, R. (Ed.), Essays in Game Theory and Mathematical Economics in Honor of Oskar Morgenstern. Princeton University Press.
Thomson, W., 2003. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. Math. Soc. Sci. 45, 249–297.
Thomson, W., 2015. Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: an update. Math. Soc. Sci. 74, 41–59.
Tijs, S.H., 1981. Bounds for the core and the τ-value. In: Moeschlin, O., Pallaschke, D. (Eds.), Game Theory and Mathematical Economics. North Holland Publishing Company.
van den Brink, R., Pinter, M., 2015. On axiomatizations of the Shapley value for assignment games. J. Math. Econ. 60, 110–114.
Yeh, C.-H., 2008. Secured lower bound, composition up, and minimal rights first for bankruptcy problems. J. Math. Econ. 44, 925–932.