4 Analysis of a Bankruptcy Problem from the Talmud

4.1 INTRODUCTION

Many times one encounters a bankruptcy situation where there are claims against a given estate and the sum of the claims against the estate exceeds its worth. In such situations one would like to know what would be a "fair" way of dividing the estate among the claimants.

Unfortunately, there is no clear-cut answer to this question. What seems fair in one case may seem less so in another. In this chapter we shall encounter several solutions, each shedding light on the "real world" and each applicable under certain circumstances.

We start with a curious method of division that has its origin in the Talmud,[1] which represents still another fair division. It involves a man who married three women and promised them in their marriage contract the sums of 100, 200, and 300 units of money to be given to them upon his death. The man died but his estate amounted to less than 600 units. The Mishna, attributed to Rabbi Nathan (tractate Ketubot 93a), treats the cases in which the estate was worth 100, 200, and 300 units of money. The recommendation in the Mishna is given in the following table.

Claims \ Estate	100	200	300
100	33 1/3	50	50
200	33 1/3	75	100
300	33 1/3	75	150

This recommendation of Rabbi Nathan seems strange. Why equal division if the estate is small? Why proportional division if the estate is worth 300 units? Most strangely, how did Rabbi Nathan reach

[1] An ancient document that forms the basis for Jewish religious, criminal, and civil law. It consists of the *Mishna*, which is its core, and the *Gemara*, which discusses the Mishna and expands on it. The Mishna was put into definitive form about 1800 years ago and the Gemara was sealed about 200 years later.

the division for the case in which the estate is worth 200? Above all, what should the rule be if the worth of the estate were different and if there were more widows?

Indeed, for many years this passage was not understood, and different rules of division were adopted by different rabbinic scholars. Some thought that this division reflected special circumstances whose description was neglected. Another thought that there was a spelling mistake. The wording of the Talmud itself suggests that this recommendation was not adopted, and that a different law was applied. One important rabbinic scholar, Hai Gaon, expressed the opinion that there might be some relation between this rule and the rule for dividing a garment between two claimants (see Section 4.2). However, Rabbi Hai Gaon did not explain the relation, and eventually retracted his opinion.

Despite myriad discussions among various scholars, no solid explanation was found until quite recently. Two game theorists, R. J. Aumann and M. Maschler, examined the rule. They decided to translate the three bankruptcy problems into game models and see if known solution concepts would yield the results stated in the Mishna. To their surprise, they found that one solution concept, called the *nucleolus*, gave precisely the numbers of the above table. It seemed that, finally, an explanation of Rabbi Nathan's recommendation had been found. There was only one "minor" problem: the nucleolus was invented by D. Schmeidler[2] in 1969. It was absolutely inconceivable that Rabbi Nathan knew what the nucleolus was.[3] There had to be another explanation for the numbers in the table. A hint was found in a paper by the game theorist A. I. Sobolev, who provided a system of axioms that characterize the nucleolus.[4] One of these axioms, called *consistency*, was the right clue.

[2] Schmeidler, D. 1969. "The nucleolus of a characteristic function game," *SIAM Journal of Applied Mathematics* 17: 1163–70

[3] A description of the nucleolus is beyond the scope of this book.

[4] Sobolev, A. I. 1975. "The characterization of optimality principles in cooperative games by functional equations," in Vorobiev, N. N. (ed.), *Matematicheskie Metody v Socialnix Naukax* 6. Academy of Sciences of the Lithuanian S. S. R., Vilnius, pp. 94–151

In this chapter we explain the concept of consistency and show how it yields a reasonable explanation of Rabbi Nathan's table. Moreover, it shows clearly how similar problems with more creditors and various claims can be resolved.[5]

To understand this explanation we first have to understand another, simpler Mishna rule involving a contested garment.

4.2 THE CONTESTED GARMENT

The following Mishna appears in the Talmud (tractate Bava Metzia 2a): "Two hold a garment; both claim it all. Then the one is awarded half, the other half. Two hold a garment; one claims it all, the other claims half. Then the one is awarded 3/4, the other 1/4."

We shall now discuss the claims and the decision of this Mishna. In the first case, both sides claim the whole garment and the decision establishes that in this case each claimant gets half the length of the garment.

The second case is of much greater interest to us. The one claims the whole garment and the other claims half. In this case the decision establishes that the claimant to the whole garment receives 3/4 of it and the claimant to half the garment receives 1/4.

How was this division reached? Rabbi Shlomo Yitzhaki (Rashi) interprets the decision as follows. The claimant to half the garment "concedes ... that half belongs to the other, so that the dispute revolves solely around the other half. Consequently, ... each of them receives half the disputed amount." Thus it is decided that the division shall be 3/4 and 1/4.

In this section we shall generalize the problem to other cases.

Example 1

The garment is worth 100 units of money.
One claims that his share of the garment is 50 units.
The other claims that his share of the garment is 80 units.
How should they divide it?

[5] Aumann, R. J. and Maschler, M. 1985. "Game-theoretic analysis of a bankruptcy problem from the Talmud," *Journal of Economic Theory* 36: 195–213

Solution:

The claimant to 50 units of money declares in effect that he has no claim to the second 50 units, and, as far as he is concerned, the other claimant can have them. The claimant to 80 units declares that he has no claim to the remaining 20 units, and, as far as he is concerned, the first claimant can have them. Thus uncontested, 70 of the 100 units are divided. The division therefore revolves around the remaining 30 units of money, which are to be divided equally between the two. The description of the division is as follows.

Value of garment	100	
The two claims	80	50
Uncontested division	50	20
Equal division of remainder	15	15
	65	35

The claimant to 50 units gets 35 and the claimant to 80 units gets 65.

Example 2

A man has two creditors; one's claim is 300, the other's, 90. The man's estate is worth 120 units. This is a bankruptcy problem. We shall solve it according to the "contested-garment" principle.[6]

Estate	120	
Claims	90	300
Uncontested division	0	30
Equal division of remainder	45	45
	45	75

[6] We are taking the position that any claim greater than the estate should be truncated to the size of the estate since there is nothing more to divide.

Answer:

The claimant to 300 units gets 75 and the claimant to 90 units gets 45.

A new element appears in Example 2. One of the debts exceeds the total amount available for distribution. It is worth noting that the creditors address their claims to the debtor and not to each other. The claimant to 90 units has no claim on the remaining 30 units. As far as he is concerned, those 30 units can be paid to the other creditor. On the other hand, the claimant to 300 units in effect claims the entire estate. Unfortunately for him, he cannot claim more than that amount, because there is no additional property. As far as he is concerned, there is no money left that he does not claim, and so, from his standpoint, there is nothing left for the other claimant, which explains the 0 that appears in the column of the claimant to 90 units.

Mathematical generalization:

The estate is E.

The creditors claim d_1 and d_2.

$d_1 + d_2 > E$; otherwise there is nothing to prevent full repayment of the debt.

Division of the estate is as follows.

Estate	E	
Claims	d_1	d_2
Uncontested division	$(E - d_2)_+$	$(E - d_1)_+$
Equal division of remainder	$\frac{E-(E-d_1)_+-(E-d_2)_+}{2}$	$\frac{E-(E-d_1)_+-(E-d_2)_+}{2}$
	$\frac{E-(E-d_1)_++(E-d_2)_+}{2}$	$\frac{E+(E-d_1)_+-(E-d_2)_+}{2}$

Explanation: The plus sign (+) in the expression $(E-d_1)_+$ or $(E-d_2)_+$ means that the expression has a value of zero if $E-d_1 < 0$ or $E-d_2 < 0$.

4.3 EXERCISES

1. A garment is worth 150 units of money. One claims 75 units, the other claims 100 units. How should they divide the garment, according to the contested-garment principle?

2. A garment is worth 200 units. One claims 120 units, the other claims 180 units. How should they divide the garment, according to the contested-garment principle?

3. A man goes bankrupt and his entire estate at the time of bankruptcy is worth 200 units. The man has two creditors; one's claim is 300 units, the other's, 200 units. How should they divide the estate between them, according to the contested-garment principle?

4. A man goes bankrupt and his entire estate at the time of bankruptcy is worth 300 units. The man has two creditors; one's claim is 250 units, the other's, 130 units. How should they divide the estate between them, according to the contested-garment principle?

5. A man dies, leaving an estate worth 500 units. The deceased has two creditors; one's claim is 400 units, the other's, 300 units. The division of the estate between them is as follows.

Estate	500	
Claims	300	400
Division of estate	150	350

Is this division made according to the contested-garment principle? If not, divide the estate according to the contested-garment principle.

6. A man dies, leaving an estate worth 200 units. The deceased has two creditors; one's claim is 100 units, the other's, 150 units. What should

the return on their claims be according to the contested-garment principle?

7. An estate is divided as follows; check whether the division is made according to the contested-garment principle.

Estate	400	
Claims	200	350
Division of estate	125	275

4.4 A PHYSICAL INTERPRETATION OF THE CONTESTED-GARMENT PRINCIPLE

In this section, we construct a set of vessels that imitate the shares of the creditors according to the contested-garment principle, following Kaminski.[7] Consider, for example, an estate with two claims: 100 and 200. Then imagine two vessels of differing sizes, representing these two claims, into which we pour fluid representing the estate. As shown in Diagrams 1–4, each vessel is composed of two parts connected by a narrow neck. The volume of each part of a vessel is equal to half the claim of the corresponding creditor. The two vessels are connected by a narrow pipe. We assume that the volumes of the necks and the pipe are negligible and considered zero. They serve merely to transfer liquid. We take care that the base areas of the two vessels are equal and that their heights are also equal. Since the claims d_1 and d_2 in this diagram satisfy $d_1 < d_2$, we achieve equal height by constructing a longer neck for the first vessel.

We represent the estate E as a fluid whose volume is equal to E. We pour this fluid into one of the vessels and note that the fluid will stay in the vessels, because $E \leq d_1 + d_2$.

The fluid (estate) that has been poured into one of the vessels now makes its way through the narrow connecting passage into the

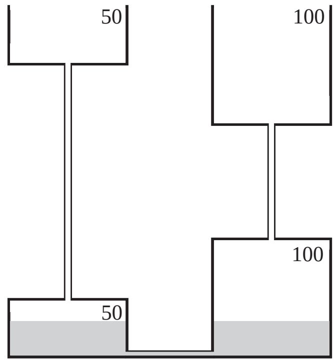
[7] Kaminski, M. M. 2000. "Hydraulic rationing," *Mathematical Social Sciences* 40: 131–55

other vessel, ultimately reaching the same level in the two vessels. This simple physical phenomenon is known as "water seeks its own level." We submit that *the amount of fluid in each of the two vessels will then be precisely what the creditor that corresponds to the vessel is entitled to, under the contested-garment principle*. We shall call this the Rule of Linked Vessels.

Let us look at some specific examples.

1. The estate is 80 and the debts are 100 and 200.

Diagram 1



The fluid that has been poured into one of the vessels makes its way through the narrow connecting passage into the other vessel, ultimately reaching the same level in the two vessels.

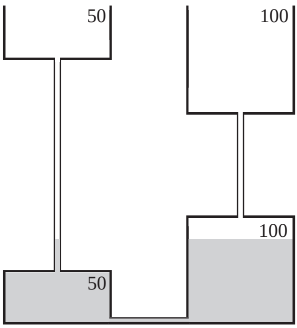
If we divide the estate of 80 between two creditors of 100 and 200 according to the contested-garment principle it will be:

Estate	80	
Claims	100	200
Uncontested sum	0	0
Contested sum	40	40
Division of estate	40	40

This is exactly what is in the two vessels.

2. The estate is 140 and the debts are 100 and 200.

Diagram 2

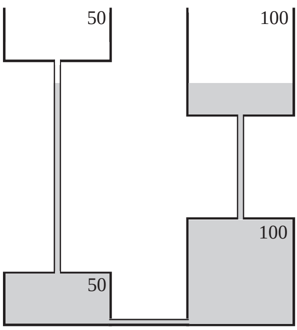


We pour the fluid (estate) and in this case the bottom part of the smaller vessel is full but the fluid does not reach the top part (see diagram). In this case the estate is more than the smaller claim but less than the bigger. When we pour a volume of 140 into the vessels, one vessel will be half filled and the other will be occupied by 90 units of fluid, as we see in the diagram. Computation of the division of E among the creditors is provided below, and we see that *it corresponds exactly to the diagram*.

Estate	140	
Claims	100	200
Uncontested sum	0	40
Contested sum	50	50
Division of estate	50	90

3. The estate is 180 and the debts are 100 and 200.

Diagram 3



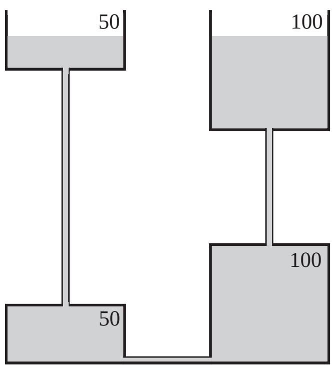
We pour the fluid and in this case, too, the bottom part of the smaller vessel is full but the fluid does not reach the top part. The estate is more than the smaller claim but still less than the bigger one. Here, the fluid occupies only half of the small vessel and 130 units of the other vessel (for a total of 180 units).

The division according to the contested-garment principle *corresponds to the diagram* as the following calculation shows.

Estate	180	
Claims	100	200
Uncontested sum	0	80
Contested sum	50	50
Division of estate	50	130

4. The estate is 240 and the debts are 100 and 200.

Diagram 4



In this case the fluid reaches the top part of both vessels. The sum of the debts is equal to 300 and the estate is equal to 240. There is a shortage of 60 units of fluid which are represented as empty parts of 30 units in each vessel. This shows that both upper halves of the vessels will be filled with fluid.

Computation of the division of the estate in accordance with the contested-garment principle, shown below, *corresponds exactly to the diagram*.

Estate	240	
Claims	100	200
Uncontested sum	40	140
Contested sum	30	30
Division of estate	70	170

These examples illustrate the fact that *for two creditors the construction of the vessels corresponds exactly to the division of an estate E in accordance with the contested-garment principle between two creditors whose claims* d_1 *and* d_2 *satisfy* $d_1 + d_2 \geq E$.

This correspondence works in two directions:

1. If we pour the fluid into the vessels and let it seek its own level, the amount of fluid in each vessel will be equal to the amount of fluid prescribed by the contested-garment principle.
2. If we disconnect the vessels and pour into each separately the amount of fluid prescribed by the contested-garment principle and then reconnect the vessels, the fluid will not flow from one vessel into the other, since it will already have reached the same height in both vessels.

4.5 EXERCISES

1. An important way of dividing an estate E among n creditors who claim $d_1, d_2, \ldots, d_n$ is to divide E among the creditors in proportion to their debts; namely, creditor i will get

$$\frac{d_i}{d_1 + d_2 + \ldots + d_n} \cdot E$$

Describe a set of vessels and their links that illustrate such a division.

2. A company is owned by three shareholders. The first shareholder owns preferred shares whose nominal value total d_1 and the other two own regular shares whose nominal values are d_2 and d_3. In case of bankruptcy, worth E of the company is distributed to the owners

according to the following rule. First, the first owner gets the nominal value of his shares, as long as $E > d_1$. Otherwise, he gets E. The rest, if any remains, is distributed to creditors 2 and 3 in proportion to their shares d_2 and d_3. Construct vessels that demonstrate how any E satisfying $E \leq d_1 + d_2 + d_3$ is divided.

3. Given an estate E and creditors claiming d_1 and d_2, $d_1 + d_2 \geq E$. Prove that the vessel construction always yields the same division as the contested-garment principle. Hint: We provided four examples above. Construct a general proof for the four examples.

4.6 A BANKRUPTCY PROBLEM FROM THE TALMUD

The Mishna tells of a man with three wives who in their marriage contracts are bequeathed sums of 100, 200, and 300 dinars, respectively. According to the law, these sums are to be paid out to the women when their husband dies. Unfortunately, the husband dies and it turns out that his estate totals less than 600. How should the estate be divided among the widows? The Mishna of Rabbi Nathan discusses three cases:

(i) The estate is 100;
(ii) The estate is 200;
(iii) The estate is 300.

His ruling is presented in the following table:

Claims \ Estate	100	200	300
100	33⅓	50	50
200	33⅓	75	100
300	33⅓	75	150

According to the table, there is equal division among the widows when the estate is 100, there is proportional division among the widows when the estate is 300, but the division is by no means clear when the estate is 200: 50 units to the widow with the marriage contract for 100 and 75 units to each of the other two widows.

Let us consider, for example, the division of the estate among the widows in the second case, where the estate is 200.

Claims \ Estate	200
100	50
200	75
300	75

Let us choose any two widows: the first and the third, for example. The two together get 125 from Rabbi Nathan. What happens if they divide this sum according to the contested-garment principle?

Estate	125	
Claims	100	300
Uncontested division	0	25
Equal division of remainder	50	50
	50	75

According to the contested-garment principle, the claimant to 100 dinars should get 50 and the claimant to 300 dinars should get 75. *Those are the precise amounts Rabbi Nathan specified for the widows!*

Now let us check the division of the estate between the widows with marriage contracts of 200 and 300. The two together get 150 from Rabbi Nathan. According to the contested-garment principle:

Estate	150	
Claims	200	300
Uncontested division	0	0
Equal division of remainder	75	75
	75	75

According to the contested-garment principle, they should each get 75. *Those are the precise amounts Rabbi Nathan specified for the widows!*

A similar calculation shows that if the contested-garment principle is applied to the amount that the widows with marriage contracts for 100 and 200 received together, then they get the precise amounts Rabbi Nathan specified for them (verify this!).

We showed that the division of the estate (50, 75, 75) is *consistent with the contested-garment principle*. Any two widows who share the amount distributed to them in accordance with the contested-garment principle will discover that they get precisely what Rabbi Nathan gave them to begin with.

The remaining cases presented in the table (p. 177) are also consistent with this principle. (In the exercises below you will be asked to verify this.)

It will be proved in Section 4.8 that these are the only numbers consistent with the contested-garment principle. Suppose someone proposes to divide an estate of 200 as (40, 60, 100). Let us check what amount is received by the first widow and the second widow. The two together get 100. Suppose now that the widows are strong believers in the contested-garment principle. Together they received 100. One claims 100 and the other claims 200 units. How should they divide the money that Rabbi Nathan allocated to them? According to the contested-garment principle, they ought to get the following amounts:

Estate	100	
Claims	100	200
Uncontested division	0	0
Equal division of remainder	50	50
	50	50

According to the contested-garment principle, they should get (50,50). Therefore, the first widow will not agree to the proposal above and she will ask for more. In other words, the proposal is *not consistent with the contested-garment principle*.

Let us now check the division of the estate between the second widow and the third widow. According to the proposal above, the two together get 160. According to the contested-garment principle, they ought to get the following amounts:

Estate	160	
Claims	200	300
Uncontested division	0	0
Equal division of remainder	80	80
	80	80

Thus, according to the contested-garment principle, they ought to get (80,80). In this case, therefore, the third widow will not agree to the proposed sum and she will oppose it. Thus the widows will oppose the proposal and it will not be implemented. Every time there is a proposal to divide the estate differently than (50,75,75), there will be at least one pair of widows who will find the proposal inconsistent with the contested-garment principle. Only the division (50,75,75) is consistent with the contested-garment principle for each of the three pairs of widows.

4.7 EXERCISES

Note to Exercises 3, 4, 5, and 9: To verify that a solution is consistent with the contested-garment principle, one has to check all the pairs. To conclude that the solution is not consistent with the contested-garment principle, it is enough to find one pair for which the solution is not consistent.

1. An estate is worth 100 units and the claims are 100, 200, and 300 units. Check whether the decision of Rabbi Nathan for each pair of widows is consistent with the contested-garment principle.

2. An estate is worth 300 units and the claims are 100, 200, and 300 units. Check whether the decision of Rabbi Nathan for each pair of widows is consistent with the contested-garment principle.

3. An estate is worth 300 units and the claims are 100, 200, and 300 units. There is a proposal to divide it $(80, 90, 130)$ between three widows. Check whether the sum that every pair of widows receives is different from the sum consistent with the contested-garment principle. (See note above.)

4. An estate is worth 300 units and the claims are 100, 200, and 300 units. There is a proposal to divide it $(80, 100, 120)$ between three widows who claim 100, 200, and 300 units. Check whether there is a pair of widows for whom there is no difference between dividing the estate according to this proposal and dividing the estate according to the contested-garment principle. (See note above.)

5. An estate is worth 200 units and the claims are 100, 200, and 300 units. There is a proposal to divide it $(50, 70, 80)$ between three widows who claim 100, 200, and 300 units. In this case there is only one pair of widows who will oppose the proposal. Which pair is it? (See note above.)

6. A man with an estate worth 400 units goes bankrupt. There are three creditors with claims of 150, 200, and 350 units, respectively. There is a proposal to divide the estate $(75, 100, 225)$ between the creditors. Check whether this proposal is consistent with the contested-garment principle for every pair of creditors.

7. A man with an estate worth 120 units goes bankrupt. There are three creditors with claims of 50, 90, and 130 units, respectively. Is the proposal to divide the estate $(25, 45, 50)$ consistent with the contested-garment principle for every pair of creditors?

8. A man with an estate worth 500 units goes bankrupt. There are three creditors with claims of 150, 250, and 300 units, respectively. The division of the estate is $(100, 150, 250)$. In this case one pair of creditors will get their share of the division according to the contested-garment principle. Which pair is it?

9. A man with an estate worth 200 units goes bankrupt. There are four creditors with claims of 50, 100, 150, and 200 units, respectively. Check whether the proposal to divide the estate $(25, 50, 62\frac{1}{2}, 62\frac{1}{2})$ is consistent with the contested-garment principle for every pair of creditors. (See note above.)

10. A man with an estate of 500 units goes bankrupt. There are four creditors with claims of 100, 150, 250, and 350 units, respectively. Is the proposal to divide the estate $(75, 125, 150, 150)$ consistent with the contested-garment principle for every pair of creditors?

4.8 EXISTENCE AND UNIQUENESS

In the previous sections we studied a specific example from the Talmud and learned that it is a solution that is consistent with the contested-garment principle. Three questions now come to mind.

1. Does there always exist a solution that is consistent with the contested-garment principle? For example, perhaps there is an estate that is bankrupt and its worth has to be shared by 15 creditors whose claims are such that no matter how they share the estate, there will always be two creditors who will find out that what they were offered does not satisfy the contested-garment principle.

2. Is the solution always unique? For example, perhaps there is an eight-person bankruptcy case in which there are two ways to share the estate and both are consistent with the contested-garment principle.

3. What is the solution? Take a five-person bankruptcy situation, with an estate and debts of a given size. How can we find exactly what share each creditor should get that is consistent with the contested-garment principle?

In this section we shall answer questions 1 and 2 affirmatively. The last question will be addressed in Section 4.9.

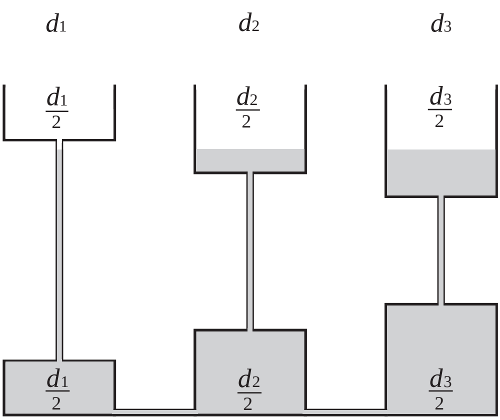
Theorem:
For any number of claimants and an estate in a bankruptcy situation, there always exists a share of the bankrupt estate that is consistent with the contested-garment principle.

Proof: Let E be an estate and let $d_1, d_2, \ldots, d_n$ be the non-negative claims against the estate demanded by creditors $1, 2, \ldots, n$. To be a bankruptcy situation, it must be that

$$E \leq d_1 + d_2 + \ldots + d_n.$$

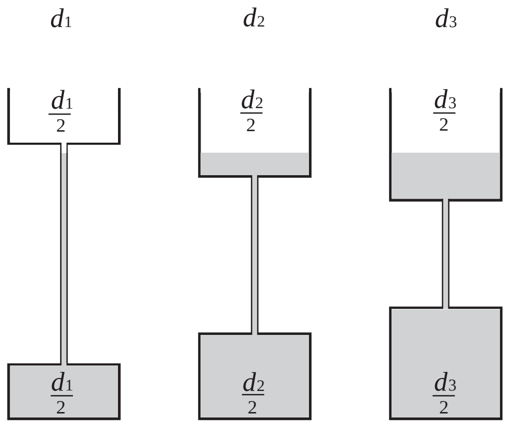
To see this, construct n vessels as described in Section 4.4 and connect them as shown in Diagram 5 (done for the case $n = 3$).

Diagram 5



Make sure that the heights as well as the base areas of the three vessels are equal. Now pour E units of fluid into the vessels. The fluid will not overflow the vessels, because $E \leq d_1 + d_2 + \ldots + d_n$. Let the fluid settle according to the law of "water seeks its own level." Disconnect the pipes connecting the vessels. You get the separate vessels as shown in Diagram 6.

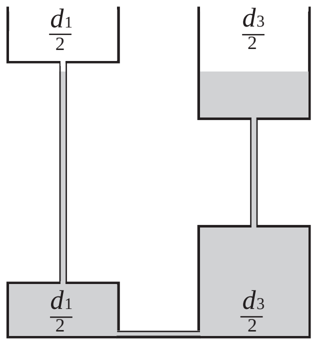
Diagram 6



We claim that the amount of fluid in each vessel represents the share of each corresponding creditor. Note that the fluid reaches the same height in all three vessels.

Take any two vessels i and j and connect them (Diagram 7).

Diagram 7

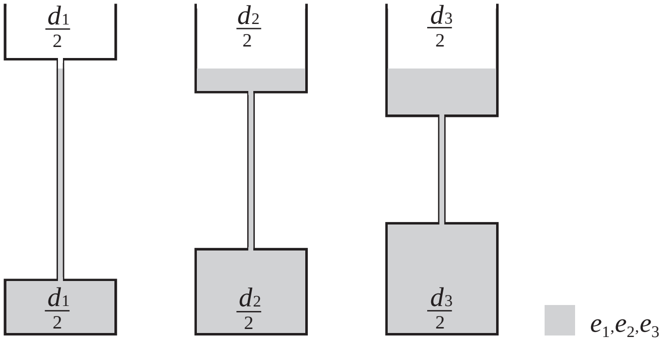


Notice that the fluid does not flow from one vessel into the other, because the height of the fluid in both vessels is already the same. Thus, the fluid in vessels i and j obeys the Rule of Linked Vessels. This proves that the share that we propose is consistent with the contested-garment principle.

Theorem:
There is only one way to share the estate E with creditors $d_1, d_2, \ldots, d_n$ that is consistent with the contested-garment principle and it is the one described in the previous theorem.

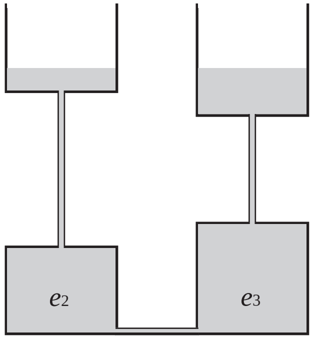
Proof: Let $e_1, e_2, \ldots, e_n$ be a share of E that is consistent with the contested-garment principle. Consider the vessels as before but do not connect them as of yet (Diagram 8).

Diagram 8



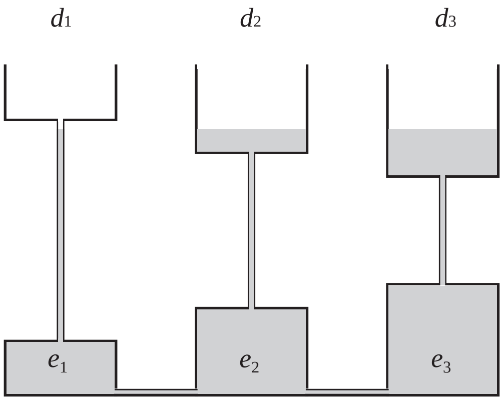
Pour amounts of fluid $e_1, e_2, \ldots, e_n$ into vessels $1, 2, \ldots, n$. Take any two vessels i and j and connect them (Diagram 9).

Diagram 9



The fluid will be at the same level because the solution $(e_1, e_2, \ldots, e_n)$ is consistent with the contested-garment principle. This is true for every pair of vessels, so the fluid is at the same height in all of them. (Explain.) Now connect all the vessels and you see that no fluid will flow from one vessel into the others (Diagram 10 for the case $n = 3$).

Diagram 10



This shows that a consistent contested-garment principle must be as described in the previous theorem.

4.9 DIVISIONS CONSISTENT WITH THE CONTESTED-GARMENT PRINCIPLE

At this point one may ask what the recommended division should be if the estate is not necessarily worth 100, 200, or 300 units. In this section we shall present a law for the division of an estate that is worth less than the sum of its claims. We shall discuss the case of three widows with marriage contracts of 100, 200, and 300 when the values of the estate are different from these. From the rule that we shall establish it will be clear how to extend it to any claims and any number of creditors.

The following table (Table 1) describes three cases where the law is enforced, when the estate has an upper bound of 300.

Table 1:

Claims \ Estate		150		250		300
100	α	50	50	50	50	50
200	α	50	50+β	100	100	100
300	α	50	50+β	100	100+γ	150

Explanation: When the estate is small, it is divided equally among the widows (first column from left). Every unit of money that comes from the estate is divided equally among the widows. That is, it is divided equally among them until the first widow obtains half of her marriage contract (second column from left). At this stage the estate is 150.

From this stage on, each additional unit of money is divided equally between the second widow and the third widow (third column from left). That is, each additional unit is divided equally between the second widow and the third widow until the second widow obtains half of her marriage contract. At this stage the estate is 250. From this stage on, each additional unit of money is given to the third widow only (fifth column from left). That is, each additional unit is given to the third widow until she obtains half of her marriage contract. The estate at this stage is 300.

The following table describes the enforcement of the law when the estate exceeds 300 units, but does not exceed 600 units.

Table 2:

Claims \ Estate	300		350		450		600
100	50	50	50	50	50	100–α	100
200	100	100	100	150–β	150	200–α	200
300	150	200–γ	200	250–β	250	300–α	300

Explanation: In this case we examine the losses. When the estate is 600 units (or more), there is no problem in dividing it; each widow obtains her marriage contract (first column from right). When the estate is under 600 units, the widows incur equal losses (second column from right). That is, they incur equal losses until the first widow loses half of her marriage contract. At this stage the estate is 450 units. From this stage on, each additional loss is divided equally between the second widow and the third widow only (fourth column from right), until the second widow loses half of her marriage contract. The estate at this stage is 350 units. From this stage on, only the third widow incurs losses (sixth column from right), until she loses half of her marriage contract and the estate is 300 units.

Now we shall check the second column in Table 2, when $\gamma = 15$.

Claims \ Estate	335
100	50
200	100
300	185

We conclude by verifying that this division indeed obeys the contested-garment principle.

150		235		285	
100	200	100	300	200	300
0	50	0	135	0	85
50	50	50	50	100	100
50	100	50	185	100	185

This law can easily be generalized to cases where the claims are different and the number of claimants is greater.

Example 1

There are four creditors with claims of 120, 140, 200, and 250 units of money, respectively. The estate that is divided to cover the debt is worth only 300 units. How should they divide the estate according to the law described above?

Solution: The total amount of claims in this case is 710 units. The estate supposed to cover the debt is worth 300 units; i.e., it is less than half the total amount of claims. It will be helpful, therefore, to consult Table 1.

Let us complete the table as described above, until we exceed the amount of 300 units.

Claims \ Estate	240	270	330
120	60	60	60
140	60	70	70
200	60	70	100
250	60	70	100

We exceed the amout of 300 units when we divide each additional unit equally between the two last claimants. Thus, we shall deduct the surplus amount, when it is evenly divided between the two.

The division obtained is $(60, 70, 85, 85)$.

Exercise: Check whether the amount received by the first claimant and the third claimant in this division is consistent with the contested-garment principle.

Example 2

There are four creditors with claims of 120, 140, 200, and 250 units of money, respectively. The estate to be divided is 420 units. How can the estate be divided in a way that is consistent with the contested-garment principle?

Solution: In this case, the estate to be divided is greater than half the amount of the debts ($710 \div 2 = 355$). We are concerned, therefore, with the losses and it will be helpful to consult Table 2, which we shall complete from right to left:

Claims \ Estate	380	440	470	710
120	60	60	60	120
140	70	70	80	140
200	100	130	140	200
250	150	180	190	250

In the last column we have obtained less than the total amount at our disposal to divide; i.e., we have deducted too much from the third claimant and the fourth claimant. We must divide 420; hence it is necessary to add 40 units, which are divided equally between the last two claimants. The requested division, therefore, is (60, 70, 120, 170).

For example, let us check whether the second claimant and the fourth claimant have received amounts consistent with the contested-garment principle.

$$
\begin{array}{cc}
\multicolumn{2}{c}{240} \\
\hline
140 & 250 \\
\hline
0 & 100 \\
70 & 70 \\
\text{—} & \text{—} \\
70 & 170
\end{array}
$$

Thus, our check shows consistency with the contested-garment principle.

Summary: In this section we introduced a procedure for the division of an estate among creditors. Implementation of this procedure requires partial completion of a table – in terms of profits, if the estate is less than half the amount of the claims, and in terms of losses, if the estate is greater than half the amount of the claims. One completes the table, until one gets the correct division. The reader can ascertain that the procedure described above imitates the liquid poured into the vessels. This proves the following theorem:

Theorem:
The procedure for the division of an estate described above results in the end in a division consistent with the contested-garment principle for every pair of creditors. By any other division there will be at least one pair of creditors for whom the amounts received are not consistent with the contested-garment principle.

On the basis of this theorem, every outcome of this procedure is *contested-garment-consistent*.

4.10 EXERCISES

1. A man dies, leaving an estate worth 500 units of money. The deceased has three widows with marriage contracts of 100, 200, and 300 units, respectively. Divide the estate among the widows, such that the division is contested-garment-consistent.

2. Divide an estate worth 300 units among three widows with claims of 50, 100, and 200 units, respectively, such that the division is contested-garment-consistent.

3. Divide an estate worth 230 units among four widows with claims of 50, 100, 150, and 200 units, respectively, such that the division is contested-garment-consistent.

4. Divide an estate worth 350 units among four widows with claims of 80, 120, 160, and 200 units, respectively, such that the division is contested-garment-consistent.

5. Divide an estate worth 800 units of money among six widows with claims of 50, 100, 150, 200, 250, and 300 units, respectively, such that the division is contested-garment-consistent.

6. Divide an estate worth 400 units among five widows with claims of 70, 100, 160, 220, and 300 units, respectively, such that the division is contested-garment-consistent.

7. Check whether the (25, 75, 125, 175) division of an estate worth 400 units among four widows with claims of 50, 100, 150, and 200 units, respectively, is contested-garment-consistent.

8. Check whether the (50, 100, 150, 200, 200) division of an estate worth 700 units among five widows with claims of 75, 125, 200, 250, and 300 units, respectively, is contested-garment-consistent.

9. The following table presents divisions of an estate in various amounts for four creditors. (The upper row represents the different estates and the left-hand column represents the different claims.) Check whether all the divisions in the table are contested-garment-consistent. Indicate which divisions are not contested-garment-consistent.

Claims \ Estate	100	150	200	300	400
50	25	37.5	25	25	25
100	25	37.5	50	50	75
200	25	37.5	62.5	100	150
300	25	37.5	62.5	125	150

4.11 CONSISTENCY

Let us return to Example 2 in Section 4.9. This example involves four creditors with claims and a division of the estate among them as follows.

	420
120	60
140	70
200	120
250	170

The question is, assuming the amount received by three of the four claimants (say, the first, third, and fourth, who together get 350 units) is divided according to the contested-garment principle, is the same division obtained as when the estate is divided among three of the four claimants?

The answer to this question is positive and can be proved in two ways.

First Proof: The part of the estate paid by the three claimants is the same as in the original problem and in the three-person problem, namely, 350 units. The division of this sum is consistent with the contested-garment principle for any pair of players, and, in particular, for any pair in the three-person problem. Thus, the solution for the four-person problem, restricted to the three-person problem, is indeed consistent with the contested-garment principle.

Second Proof: We construct the appropriate table until we have two adjacent columns: one with the estate above 350 units and one with the estate below 350 units.

	310	390	570
120	60	60	120
200	100	140	200
250	150	190	250

In this range deduction takes place only between the last two claimants. From 350 units we still have to deduct 40 units, to be divided equally between these claimants. We get the division $(60, 120, 170)$, which is exactly what all three creditors received in the four-person problem.

The above example is a special case of the following theorem:

Theorem:
If a set of creditors divides an estate according to the contested-garment principle, then each subset that divides the amount that its members obtained in the original division, while respecting the original claims and according to the contested-garment principle, will get precisely the same division that they obtained in the original division.

We can summarize the theorem as follows:
A division according to the contested-garment principle is a division that is consistent for any number of its participants (and not just for any two participants).

4.12 EXERCISES

1. (1) Divide an estate worth 550 units of money among four creditors with claims of 50, 150, 200, and 300 units, respectively, according to the contested-garment principle.
   (2) Check whether the total amount received by the claimants to 50, 200, and 300 units will be divided among them in the same way if it is divided among the three of them according to the contested-garment principle.
2. (1) Divide an estate worth 400 units among six creditors with claims of 50, 80, 100, 140, 200, and 250 units, respectively, according to the contested-garment principle.
   (2) Check whether the total amount received by the claimants to 80, 140, 200, and 250 units will be divided among them in the same way if it is divided among the four of them according to the contested-garment principle.
3. (1) Divide an estate worth 900 units among six creditors with claims of 100, 150, 200, 260, 300, and 320 units, respectively, according to the contested-garment principle.
   (2) Check whether the total amount received by the claimants to 100, 200, and 300 units will be divided among them in the same way if it is divided among the three of them according to the contested-garment principle.

4.13 RIF'S LAW OF DIVISION

The Rif (Rabbi Yitzhak Alfasi) proposed another law of division, later adopted by Rambam (Rabbi Moshe Ben Maimon). According to this law, every unit of money is divided equally among all claimants, until the claimant with the smallest claim gets his full amount. Each additional unit of money is divided equally among the remaining creditors, until the claimant with the smallest claim at this stage gets his full amount, and so on.

Example:

Claims \ Estate		300		500		600
100	α	100	100	100	100	100
200	α	100	$100+\beta$	200	200	200
300	α	100	$100+\beta$	200	$200+\gamma$	300

Explanation: Equal division occurs until the first claimant receives her claim (the 300-column). Then, the other claimants receive additional equal amounts until the second claimant receives her claim (the 500-column). At that stage, the last claimant receives the remainder of the estate, but not more than his claim (the 600-column).

Suppose now that the estate is 350 units. To divide it, we construct the above table up to and inclusive of the 500-column (explain) and we see that the division must be (100, 125, 125).

Is Rif's law of division consistent? That is, will every subset of claimants that divides the total received by the claimants in the original division according to Rif's law, get the same amounts? Let us check how much the first claimant and the third claimant get when the estate is 350. The two together got 225. We shall divide this sum between them according to Rif's law.

Claims \ Estate	200	225
100	100	100
300	100	125

Explanation: First, we divided 100 units for each of the two claimants. The first got his full amount. The rest was given to the second claimant.

We see that in this case there is consistency between the first and third creditors.

Exercise: Check whether there is consistency between the first and second creditors, and between the second and third creditors.

It can be proved that Rif's law of division is indeed consistent; i.e., for any subset, if we distribute among its members the total payoffs they received together in the original division according to Rif's law, the same division will be obtained.

4.14 EXERCISES

1. Divide an estate worth 275 units among four creditors with claims of 50, 100, 150, and 200 units, respectively, according to Rif's law.

2. Divide an estate worth 400 units among four creditors with claims of 50, 100, 150, and 200 units, respectively, according to Rif's law.

3. (1) Divide an estate worth 790 units among five creditors with claims of 100, 150, 200, 250, and 300 units, respectively, according to Rif's law.
   (2) Check whether the division according to Rif's law is consistent, say, for a group of creditors with claims of 150, 250, and 300 units, respectively.

4. (1) Divide an estate worth 400 units among five creditors with claims of 40, 60, 80, 120, and 150 units, respectively, according to Rif's law.
   (2) Check whether the division according to Rif's law is consistent, say, for a group of creditors with claims of 40, 60, 80, and 150 units, respectively.

4.15 PROPORTIONAL DIVISION

In the world of finance it is customary to divide an estate in proportion to the investments.

Example:

Four partners founded a company that later closed due to financial difficulties. We divide its market value – $555,000 – among the partners proportionally to their shares in the company, which are 40, 60, 120, and 150, respectively.

When the total number of shares in the company is $40 + 60 + 120 + 150 = 370$:

The first gets: $\frac{555 \cdot 40}{370} = 60$
The second gets: $\frac{555 \cdot 60}{370} = 90$
The third gets: $\frac{555 \cdot 120}{370} = 180$
The fourth gets: $\frac{555 \cdot 150}{370} = 225$

Question: Is the law of proportional division consistent?

Answer: Let's check for the first three shareholders, who received $60 + 90 + 180 = 330$. We divide this amount proportionally among them. When their shares are $40 + 60 + 120 = 220$, then:

The first gets: $\frac{330 \cdot 40}{220} = 60$

The second gets: $\frac{330 \cdot 60}{220} = 90$

The third gets: $\frac{330 \cdot 120}{220} = 180$

Thus, the amounts the shareholders obtain are precisely those they got in the original division.

It is easy to prove that the proportional division is also a consistent solution. Any subset of players who examine the amounts distributed to them will find them proportional to their claims.

4.16 O'NEILL'S LAW OF DIVISION

O'Neill presents another interesting law.[8] Consider, for example, a case where the estate is worth 250 units and the claims are 100, 200, and 300 units, respectively. The creditors rush to the bank or to wherever the estate is disbursed. The first to arrive gets his claim

[8] O'Neill, B. 1982. "A problem of rights arbitration from the Talmud," *Mathematical Social Sciences* 2: 345–71

in full, because no other claims have been presented. The second to arrive gets his claim in full or in part, depending on the amount of money left over from the first claim, and so on. Every creditor to arrive gets his claim in full or in part, until the estate is depleted. The amount each creditor gets depends, of course, on the order of arrival.

O'Neill's law proposes that, instead of holding a race, each creditor compute all he can get according to the order of his arrival, over all possible orders. The amount each creditor gets in the end will be the average of the amounts received in all possible orders.

The estate is 250:
1 claims 100:
2 claims 200:
3 claims 300:

Order of arrival \ Creditors	1	2	3
123	100	150	0
132	100	0	150
213	50	200	0
231	0	200	50
312	0	0	250
321	0	0	250
	(250,	550,	700):6 = (41⅔, 91⅔, 116⅔)

The final division is the average of the amounts, namely, $\left(41\frac{2}{3}, 91\frac{2}{3}, 116\frac{2}{3}\right)$.

Is O'Neill's law consistent?

Let us consider, say, the first and third creditors. They together received $158\frac{1}{3}$, while their claims are 100 and 300, respectively. The division according to O'Neill's law will be as follows.

The estate is $158\frac{1}{3}$:
1 claims 100:
2 claims 300:

Order of arrival \ Creditors	1	2
12	100	$58\frac{1}{3}$
21	0	$158\frac{1}{3}$
	$(100, 216\frac{2}{3}):2 = (50, 108\frac{1}{3})$	

According to the law the creditors will get $(50, 108\frac{1}{3})$, which is not the original division. We see that this law of division does not satisfy the consistency property.

Every bankruptcy problem of the kind we have discussed up until now can be translated to an $(N; v)$ game where N is the set of creditors and the coalition function is defined as:

$v(S) =$[estate minus amount of claims of creditors who are not in $S]_+$

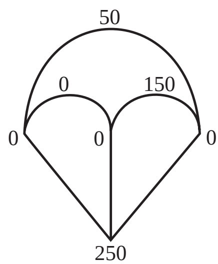
Explanation: The amount due to the individuals in S without any division is the sum that is left over from the estate after the creditors who are not in S receive their claims in full. Thus the creditors in S can guarantee themselves this amount. If the difference between the two amounts is negative, then we set $v(S) = 0$, and that is the meaning of the plus sign in the formula above.

Let us now translate the example above to an $(N; v)$ game: the estate totals 250 units and creditors 1, 2, and 3 claim 100, 200, and 300 units, respectively.

$N = \{1, 2, 3\}$

$$
\begin{aligned}
v(1) &= [250 - (200 + 300)]_+ = 0\\
v(2) &= [250 - (100 + 300)]_+ = 0\\
v(3) &= [250 - (100 + 200)]_+ = 0\\
v(1,2) &= [250 - 300]_+ = 0\\
v(1,3) &= [250 - 200]_+ = 50\\
v(2,3) &= [250 - 100]_+ = 150\\
v(1,2,3) &= [250 - 0]_+ = 250\\
v(\emptyset) &= [250 - 250]_+ = 0
\end{aligned}
$$

The game can be presented in the following figure:



We calculate the Shapley value of the game.

Orders \ Creditors	1	2	3
123	0	0	250
132	0	200	50
213	0	0	250
231	100	0	150
312	50	200	0
321	100	150	0
	(250,	550,	700):6 = $(41\frac{2}{3}, 91\frac{2}{3}, 116\frac{2}{3})$

We see that the Shapley value of this game is precisely O'Neill's solution.

4.17 EXERCISES

1. Divide an estate worth 400 units among three creditors with claims of 100, 200, and 300 units, respectively, according to O'Neill's law.

2. Divide an estate worth 500 units among four creditors with claims of 100, 150, 200, and 250 units, respectively, according to O'Neill's law.

3. There is a bankruptcy problem in which an estate is worth 500 units and the claims are 100, 300, and 400 units, respectively. Translate the problem to a coalition game and calculate the Shapley value of the game. Show that the O'Neill procedure leads to the same division.

4. There is a bankruptcy problem in which an estate is worth 700 units and the claims are 200, 250, 300, and 400 units, respectively. Translate the problem to a coalition game and calculate the Shapley value of the game. Show that the O'Neill procedure leads to the same division.

5. (1) Divide an estate worth 200 units among four creditors with claims of 50, 100, 150, and 200 units, respectively, according to O'Neill's law.
   (2) Show that the law is not consistent by checking a pair of creditors.

6. (1) Divide an estate worth 300 units among four creditors with claims of 80, 120, 200, and 280 units, respectively, according to O'Neill's law.
   (2) Translate the problem to a coalition game.
   (3) Show that O'Neill's law is not consistent by checking three creditors.

4.18 DISCUSSION

In this chapter we presented four different laws for dividing an estate among creditors when the total amount of the claims against the estate exceeds the value of the estate. Note that some of them apply to various situations in real life. For example, the proportional law applies when a company of shareholders goes bankrupt. O'Neill's "running to the bank" solution, which is also the Shapley value of an appropriate coalition function, can be understood as an a priori expectation in those cases where the players actually run to the bank and there is no way of telling in advance in what order they will arrive there. The Talmudic law of Rabbi Nathan can be considered desirable when the players want to share equally the contested part of the debts.

Can we say which solution is superior to the others? Obviously not, because each of them is considered better suited to a particular case.

Since we cannot say that any one solution is absolutely superior, which solution should we recommend when a new real-life situation arises? Even if each solution sheds light on different aspects of the case, we usually have to decide on a single solution. How does one make this choice? That is, what criteria should guide one's choice in preferring any one solution to the others in a given real-life case? We can only provide guidelines:

(a) Look at the axioms and properties that characterize each solution and see which axioms better fit the reality. For example, the requirement of consistency is sometimes appealing, and it is this requirement that gives rise to the proportional solution, the Talmudic solution of Rabbi Nathan, and others.

(b) Look at the behavior of the players in real life. For example, perhaps they do "run to the bank," in which case O'Neill's solution, which is also the Shapley value, yields an a priori expectation on the final settlement.

(c) In complex situations, offer the players a simpler problem to which they can suggest an intelligent solution, say, a two-person case, and try to learn from their choice what aspects of the simpler problem they focus on. Then generalize to the more complex real-life case.

Broadly speaking, all the chapters in this book represent attempts at reaching a decision in a conflict situation and in each of them we show the difficulties when trying to define a "superior" solution. The first chapter on matching presents a "weak" condition of stability, which nevertheless yields many matchings. One of them is best for the men and another is best for the women. The second chapter tries to reach a decision by voting and we saw that a fair voting rule is not always possible. The third chapter has probably the most successful solution. It provides a solution for an unbiased arbitrator, by supplying axioms that seem fair. However, somewhat different axioms, not covered in this book, yield different solutions. Finally, the fourth chapter, which considers the case of bankruptcy conflicts, shows that even in this simple case a superior solution cannot be defined.

In conclusion, we see that various solutions are well tailored to many real situations, but there is no single solution that fits all situations. Each solution sheds some light on the reality.

4.19 REVIEW EXERCISES

1. A man dies, leaving an estate worth 500 units of money. The deceased has two creditors; one claims 350 units and the other claims 300 units. How will the estate be divided between them according to the contested-garment principle, Rif's law of division, proportional division, and O'Neill's law of division?

2. A man with an estate worth 1000 units goes bankrupt. The bankrupt man has four creditors with claims of 200, 300, 400, and 500 units of money, respectively. Divide the estate among the creditors according to the contested-garment principle, Rif's law of division, proportional division, and O'Neill's law of division.

3. (1) Divide an estate worth 800 units among six creditors with claims of 50, 100, 150, 200, 250, and 300 units, respectively, according to the contested-garment principle, Rif's law of division, and proportional division.
   (2) Check whether the total amount received by the creditors with claims of 50, 150, 250, and 300 units will be divided according to the divisions specified in 3(1).
4. (1) Divide an estate worth 700 units among four creditors with claims of 100, 200, 250, and 350 units of money, respectively, according to the contested-garment principle, Rif's law of division, and proportional division.
   (2) Check whether the division is consistent for a group of creditors with claims of 100, 250, and 350 units, respectively, for the divisions specified in 4(1).

5. There is a bankruptcy problem in which an estate is 800 units and the claims are 200, 300, and 400 units, respectively. Translate the problem to a coalition game and calculate the Shapley value of the game. Show that the O'Neill procedure leads to the same division.