COWLES FOUNDATION FOR RESEARCH IN ECONOMICS

AT YALE UNIVERSITY

Box 2125, Yale Station
New Haven, Connecticut 06520

COWLES FOUNDATION DISCUSSION PAPER NO. 473

THE COMPUTATION OF EQUILIBRIUM PRICES: AN EXPOSITION

Herbert E. Scarf

November 14, 1977

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I. THE GENERAL EQUILIBRIUM MODEL

A demonstration of the existence of equilibrium prices for a general Walrasian model of competitive behavior necessarily makes use of some variant of Brouwer's fixed point theorem as an essential step in the argument. The strategy for calculating equilibrium prices is to render that step constructive by a numerical approximation of the fixed point implied by Brouwer's theorem or one of its alternatives. We begin this chapter with a brief review of the competitive model and the role of fixed point theorems.

The typical consumer will be assumed to have a set of preferences for commodity bundles $x = (x_1, \ldots, x_n)$ in the non-negative orthant of n dimensional space. These preferences can be described either by an abstract preference relationship, $\succsim$, satisfying a series of plausible axioms, or by a specific utility indicator $u(x)$.

It is customary to assume that, prior to production and trade, the typical consumer will own a non-negative vector of commodities w whose evaluation by means of market prices forms the basis for that consumer's income. More specifically if the non-negative vector of prices $\pi = (\pi_1, \ldots, \pi_n)$ is expected to prevail then the consumer can obtain an income

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*To appear as a chapter in the Handbook of Mathematical Economics, edited by Arrow and Intriligator. The research was supported by a grant from the National Science Foundation.

$$I = \sum_{1}^{n} \pi_i \cdot w_i$$

by the sale of his commodity bundle w. This income is then used by the consumer for the purchase of an alternative bundle of commodities in such a way as to provide him with the highest possible utility level.

Given a vector of prices for all of the commodities in the economy the consumer is faced with a maximization problem: to find a vector of commodities x which maximizes his utility function (or is maximal according to his preference relationship) subject to the budget constraint

$$\pi \cdot x \leq \pi \cdot w \ .$$

The problem faced by the consumer is clearly unchanged if the price vector π is replaced by $\lambda\pi$, with λ an arbitrary positive number. This provides us with a degree of freedom in normalizing prices in any fashion which happens to be convenient. For example prices can be normalized so as to lie on the unit sphere $\sum_{1}^{n} \pi_i^2 = 1$ (meaningless from an economic point of view, but mathematically quite useful) or on the unit simplix $\sum_{1}^{n} \pi_i = 1$. For definiteness we shall adopt the latter normalization throughout most of this chapter.

Under quite acceptable assumptions, such as continuity and strict convexity of preferences the solution to the consumer's maximization problem will be a single-valued function $x(\pi)$ which is continuous on the unit simplex and which satisfies the identity

$$\sum_{1}^{n} \pi_i x_i(\pi) \equiv \sum_{1}^{n} \pi_i w_i \ ,$$

stating that the consumer's entire income will be spent on the purchase of commodities. (This is of course not meant to rule out the possibility that the consumer may choose to save some of his current income for future purchases--an aspect of consumer behavior which can be captured by declaring that certain commodities become available in the future).

For the purpose of computing equilibrium prices the demand functions $x(\pi)$ are frequently more natural to work with than the underlying utility function or preference relationship. A general equilibrium model must be fully specified in order to obtain a numerical solution: on the consumer side this is typically done by providing a numerical or algebraic description of the functions $x(\pi)$. A few examples of some of the more familiar utility functions and their associated demand functions may be appropriate at this point.

1. $u(x) = x_1^{a_1} \dots x_n^{a_n}$, with $a_i \geq 0$, $\Sigma a_i = 1$.

It is an elementary observation that a consumer with this utility function will spend the same fraction a_i of his income on the i^{th} good, independently of relative prices. Since his income is given by $\pi \cdot w$, this implies that his demand for the i^{th} good is given by

$$x_i(\pi) = \frac{a_i \pi \cdot w}{\pi_i} .$$

(There is an insignificant technical problem that these demand functions and others which are subsequently presented exhibit a discontinuity on the boundary of the unit simplex.)

2. $u(x) = \min[x_1/a_1, \ldots, x_n/a_n]$, with $a_i > 0$.

A consumer with preferences exhibiting such strict complementarity wishes to purchase a commodity bundle proportional to the vector $(a_1, \ldots, a_n)$. His demands are therefore given by

$$x_i(\pi) = a_i \cdot \pi \cdot w / \pi \cdot a \ .$$

3. $$u(x_1, \ldots, x_n) = \left(\sum_1^n a_i^{\frac{1}{b}} x_i^{\frac{b-1}{b}} \right)^{\frac{b}{b-1}} \ , \text{ with } a_i > 0 \ , \text{ and } b \leq 1 \ .$$

This is the C.E.S. utility function which includes the two previous examples for special values of the parameter b . It will be left to the reader to verify that the demand functions $x_i(\pi)$ are given by

$$x_i(\pi) = \frac{a_i \pi \cdot w}{\pi_i^b \sum_j \pi_j^{1-b} a_j} \ .$$

A model of equilibrium will generally involve a number of consuming units--individuals, aggregations of individuals or countries involved in foreign trade--each of whom has a stock of assets w , prior to production and trade, and demands which are functions of the prevailing prices π . If the j^{th} consumer's assets are represented by w^j and his demand functions by $x^j(\pi)$, then the <u>market demand functions</u> are defined by

$$x(\pi) = \sum_j x^j(\pi) \ .$$

The market demand functions specify the total consumer demand, for the goods and services in the economy, as a function of price.

Market demand functions cannot be arbitrarily specified if they are derived from individual demand functions by the process of aggregation described above. For example if the individual demand functions are continuous, the market demand functions will be continuous as well. Moreover the market demand functions will satisfy the identity

$$\pi \cdot x(\pi) \equiv \pi \cdot w$$

(if $w = \sum w^j$) known as the Walras law--resulting from the assumption that the income available for consumer purchases is accounted for fully by the sale of privately owned assets.

The market <u>excess</u> demand functions, representing the difference between market demands and the supply of commodities prior to production are defined by

$$\xi(\pi) = x(\pi) - w \ .$$

Our analysis may be summarized as follows:

1.1. [properties] The market excess demand functions will satisfy the following conditions

1. They are defined and continuous everywhere in the positive orthant other than the origin.
2. They are homogeneous of degree zero, i.e. $\xi(\lambda\pi) = \xi(\pi)$, for any positive λ.
3. The Walras Law:

$$\pi \cdot \xi(\pi) \equiv 0 \ .$$

In this form, the Walras law has an interesting geometrical interpretation when prices are normalized to lie on the unit sphere. If the

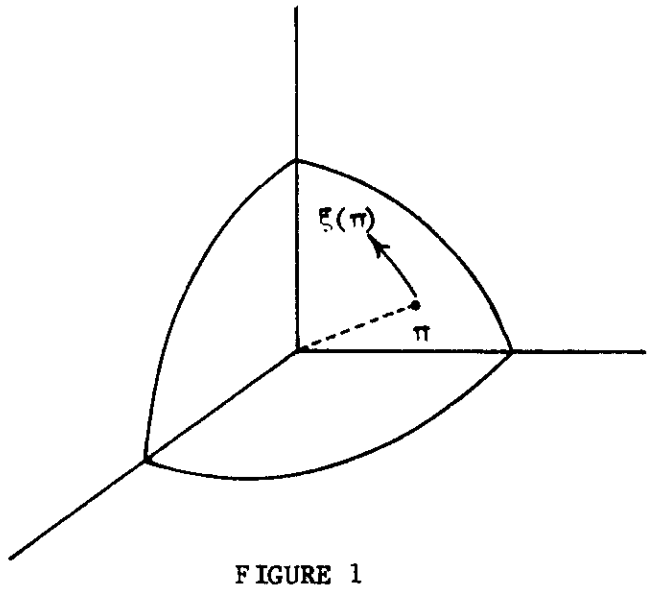

FIGURE 1

excess demand $\xi(\pi)$ is drawn as a vector originating at π, the geometric interpretation of the Walras law is that this vector is tangent to the unit sphere at π. The excess demands determine, therefore, a continuous vector field on that part of the unit sphere lying in the nonnegative orthant.

A description of the consumer side of the economy, for the purposes of the general equilibrium model, is adequately provided by the market excess demand functions. To complete the model a specification of the productive techniques available to the economy must also be given. We shall return to this point, after examining the pure trade model in which production is absent and consumers simply exchange the commodities which they initially own. For such a model the total supply is given by the vector w and the market demand by the functions $x(\pi)$. An equilibrium price vector π^* is one which equilibrates supply and demand for all

commodities. We have the following formal definition which allows for the possibility that some of the commodities have a zero price and are in excess supply.

1.2. [definition] A non-zero price vector π^* is an equilibrium for the pure trade model if $\xi_i(\pi^*) \leq 0$ for all i, and is equal to zero when $\pi_i^* > 0$.

If all of the coordinates of π^* are positive then π^* is a zero of the vector field ξ on the unit sphere. The existence of a zero for an arbitrary vector field on the sphere is a subtle topological question. It is therefore a matter of some significance, in assessing the difficulty of demonstrating the existence of equilibrium prices (and in their calculation), to ask how general a vector field can be obtained by the process of aggregating individual demand functions. This question was first studied by Sonnenschein [53], and was given a definitive answer in subsequent papers by Mantel [42], Debreu [6], and MacFadden, Mantel, Mas-Colell and Richter [39]. There are some technical difficulties on the boundary of the positive orthant, but given an arbitrary continuous vector field on the sphere and an arbitrary open subset of the positive orthant, there will be a collection of n consumers with preferences and initial holdings, whose individual demand functions sum to the preassigned vector field on the open subset. In other words, market excess demand functions are essentially arbitrary aside from the conditions described in 1.1.

Walras suggested than an equilibrium price vector could be found by a "tâtonnement" process: the continued revision of non-equilibrium prices on the basis of the discrepancy between supply and demand. If π is a price vector for which $\xi_i(\pi) \neq 0$, then a formalization of our

Each column of A represents a known technical mode of production, which can be employed at an arbitrary non-negative level. Inputs into production are represented by negative entries in a given column and outputs by positive entries. The result of using techniques 1 through k at levels $x_1, \ldots, x_k$ is a net production plan, Ax, where $x = (x_1, \ldots, x_k)$. The negative entries in Ax represent a demand for factors of production which must be contributed by consumers from their stock of initial assets w. The positive entries represent increases in the stock of other assets, which can then be distributed among the consumers. An equilibrium price vector π^* is one which equilibrates the net consumer demand $\xi(\pi^*)$ with the net supply Ax. It also induces the correct selection of productive techniques on the grounds of decentralized profit maximization. We have the following formal definition of equilibrium.

1.3. [definition] A non-zero price vector π^* and a non-negative vector of activity levels x^* are in equilibrium if

1. $\xi(\pi^*) = Ax^*$
2. $\pi^* A \leq 0$
3. $\pi^* A x^* = 0$.

The second of these conditions says that all potential activities make a non-positive profit when evaluated by means of equilibrium prices π^*, and the third condition that those activities which are actually used at equilibrium make a profit of zero. The demonstration that equilibrium prices and activity levels exist--under suitable assumptions on the matrix A --is more subtle than that of the case of pure trade, and typically makes use of Kakutani's fixed point theorem rather than Brouwer's theorem. The arguments apply equally well to a description of

production given by a closed convex cone rather than an activity analysis model.

II. BROUWER'S FIXED POINT THEOREM

Let S be a closed, bounded, convex set in n-dimensional Euclidean space, which is mapped into itself by the continuous mapping $x \to f(x)$. Brouwer's Theorem, asserts the existence of a fixed point, i.e. a point for which $\hat{x} = f(\hat{x})$. In order to illustrate the theorem we consider a few simple examples.

1. The S be the closed unit interval $[0,1]$. A continuous mapping of this interval into itself is defined by an ordinary continuous function $f(x)$ whose values also lie in the unit interval. A fixed point

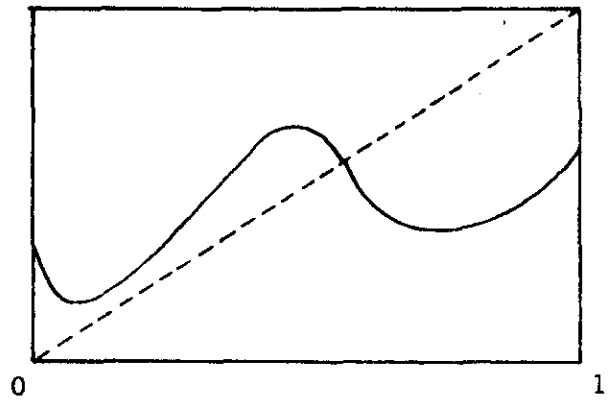

FIGURE 2

of this mapping is a point where the graph $(x, f(x))$ intersects the 45° line from the origin, or a point where $g(x) = f(x) - x$ is equal to zero. But $g(0) \geq 0$, and $g(1) \leq 0$, so that the function must be equal to zero at some point in the unit interval. Brouwer's theorem is a generalization of this well-known property of continuous functions to higher dimensions.

2. Let S be the unit simplex $\{x = (x_1, \ldots, x_n) | x_i \geq 0, \Sigma x_j = 1\}$, and let A be a square matrix with non-negative entries, whose column sums are equal to unity. The mapping

$$f(x) = Ax$$

is continuous and maps the simplex into itself. The conditions for Brouwer's theorem are satisfied and there will be a fixed point $\hat{x} = A\hat{x}$. In this particular case Brouwer's theorem implies that a non-negative square matrix with column sums of unity must have a characteristic root of 1 with an associated non-negative characteristic vector, a result which can be obtained by simpler arguments.

3. In this example we show how Brouwer's theorem can be used to demonstrate the existence of equilibrium picces for a model of exchange. We consider prices normalized so as to lie on the unit simplex S. For each π in S, $\xi(\pi)$ will be the market excess demand associated with this price. ξ is assumed to be continuous on the simplex and to satisfy the identity $\pi \cdot \xi(\pi) \equiv 0$.

Define a mapping as follows

$$f_i(\pi) = \frac{\pi_i + \max[0, \xi_i(\pi)]}{1 + \sum_j \max[0, \xi_j(\pi)]} .$$

The fact that f is continuous follows from the continuity of the market excess demand functions. It should also be clear that $f(\pi)$ is on the unit simplex if π is. Brouwer's theorem can therefore be applied and we deduce the existence of a price vector $\hat{\pi} = f(\hat{\pi})$. We shall demonstrate that $\hat{\pi}$ is an equilibrium price for the model of exchange.

If c is defined to be

$$c = 1 + \sum_j \max[0, \xi_j(\hat{\pi})] \geq 1 ,$$

then

$$c\hat{\pi}_i = \hat{\pi}_i + \max[0, \xi_i(\hat{\pi})] , \quad \text{or}$$

$$(c-1)\hat{\pi}_i = \max[0, \xi_i(\hat{\pi})] .$$

But c must be equal to 1, for if it were strictly larger this last relationship would imply that

$$\xi_i(\hat{\pi}) > 0 \quad \text{for each} \quad i \quad \text{with} \quad \hat{\pi}_i > 0 ,$$

violating the Walras Law $\sum \hat{\pi}_i \xi_i(\hat{\pi}) = 0$. If $c = 1$, however, it follows that $\sum_j \max[0, \xi_j(\hat{\pi})] = 0$, or $\xi_j(\hat{\pi}) \leq 0$ for all j.

We see that the existence theorem for exchange economies is an immediate consequence of Brouwer's theorem. It is instructive to note that the converse is also correct. The following argument shows that Brouwer's theorem follows from the existence of an equilibrium price vector for an arbitrary exchange economy.

Let $x \to f(x)$ be a continuous mapping of the unit simplex into itself. Define the functions

$$\xi_i(x) = f_i(x) - \lambda(x)x_i ,$$

where $$\lambda(x) = \sum_j x_j f_j(x) / \sum_j x_j^2 .$$

Then ξ_i are defined and continuous everywhere on the unit simplex and,

by construction, satisfy the Walras Law $\Sigma x_i \xi_i(x) \equiv 0$. From the existence theorem for competitive equilibria, we conclude that there is an $\hat{x}$ with $\xi_i(\hat{x}) \leq 0$, and equal to zero if $\hat{x}_i > 0$. But then

$$f_i(\hat{x}) \leq \lambda(\hat{x})\hat{x}_i ,$$

with equality if $\hat{x}_i > 0$. If $\hat{x}_i = 0$ we must also have equality, since otherwise $f_i(\hat{x}) < 0$, and f would not map the unit simplex into itself. It follows that

$$f_i(\hat{x}) = \lambda(\hat{x})\hat{x}_i .$$

But since $\Sigma f_i(\hat{x}) = \Sigma \hat{x}_i = 1$ it follows that $\lambda(\hat{x}) = 1$, and that $\hat{x}$ is indeed a fixed point of the mapping.

Since Brouwer's theorem and the theorem asserting the existence of equilibrium prices are equivalent to each other, any effective numerical procedure for computing equilibrium prices must at the same time be an algorithm for computing fixed points of a continuous mapping.

III. AN ALGORITHM FOR COMPUTING FIXED POINTS

In this section we shall present our first algorithm for approximating a fixed point of a continuous mapping of the unit simplex into itself. There is no loss in generality in restricting our attention to the unit simplex. If the mapping $f(x)$ is defined for a more general closed, bounded, convex set C , we embed C in a larger simplex S ,

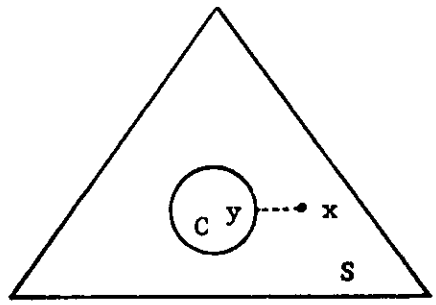

FIGURE 3

and define a mapping $g(x)$ of S into itself in the following way:

1. if $x \in C$ then $g(x) = f(x)$
2. if $x \notin C$ then $g(x) = f(y)$

where y is that unique point in C which is closest, in the sense of Euclidean distance, to x.

Since y varies continuously with x it is clear that the new mapping satisfies the conditions of Brouwer's theorem and has a fixed point $\hat{x} = g(\hat{x})$. But since $g(\hat{x}) \in C$ it follows that $\hat{x} \in C$, and is a fixed point of the original mapping f.

A simplex is defined to be the convex hull of its n vertices $v^1, \ldots, v^n$, i.e. the set of points of the form

$$x = \Sigma \alpha_j v^j \quad \text{with} \quad \alpha_j \geq 0, \quad \text{and} \quad \Sigma \alpha_j = 1.$$

The vertices are assumed to be linearly independent in the sense that each vector in the simplex has one and only one representation in the above form. The simplex, which has dimension $n-1$ in our notation, has n faces of dimension $n-2$. The face opposite the vertex v^j consists of those vectors whose representation as a convex combination of the vertices has $\alpha_j = 0$.

More generally the simplex has a number of sub-faces of lower dimension. For any subset T of $(1, 2, \ldots, n)$ there is a face of the simplex consisting of those vectors with $\alpha_j = 0$ for j in T.

We now consider a finite collection of simplices $S^1, \ldots, S^k$ contained in the large simplex S. If no restriction is placed on this collection then two of its members may have interior points in common,

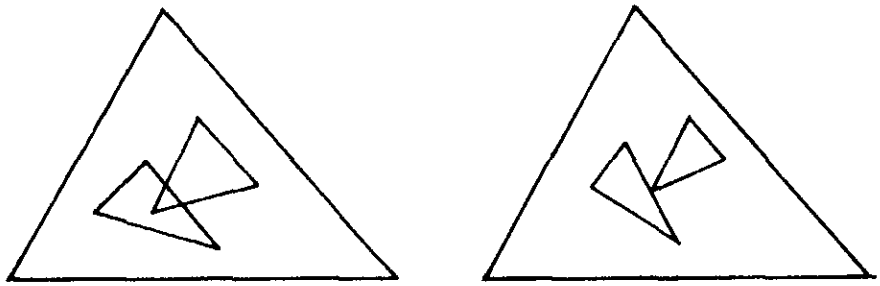

FIGURE 4

or their intersection may not consist of an entire face of one or the other of them. For the purpose of demonstrating Brouwer's theorem we shall require the collection of simplices to form what is known as a "simplicial subdivision," by ruling out these forms of intersection.

3.1. [definition] A collection of simplices $S^1, \ldots, S^k$ is called a simplicial subdivision of S if

1. S is contained in the union of the simplices $S^1, \ldots, S^k$, and 2. The intersection of any two simplices is either empty or a full face of both of them.

The following figure illustrates a typical simplicial subdivision of the simplex.

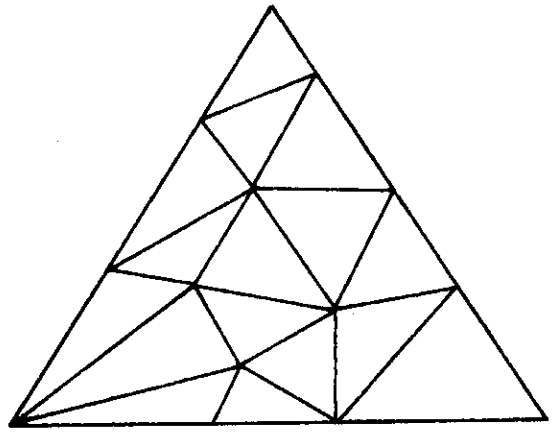

FIGURE 5

The method we shall adopt to demonstrate Brouwer's theorem and to provide an effective computational procedure for approximating fixed points makes use of a combinatorial lemma involving simplicial subdivisions of the simplex. There are alternative computational methods, noticeably those of Kellog, Li and Yorke [32], and Smale [52], which avoid simplicial subdivisions completely and use instead the methods of differential topology. It is not yet clear which of these techniques, which have greater similarities than might appear on the surface, is superior from a computational point of view.

To appreciate the combinatorial approach to Brouwer's theorem let us examine the simplest case, a continuous mapping of the unit interval into itself. A simplicial subdivision of the unit interval is given by a monotonic sequence of points, say $v^3 < v^4 < \ldots < v^k$ (I have used the notation v^1 for the lower end point of the unit interval and v^2 for the upper). In Figure 6 I have placed below each vertex v^j an integer label which is either 1 or 2 . The rule for associating the integral label is to label v^j with 1 if $f(v^j) \geq v^j$ and 2 if

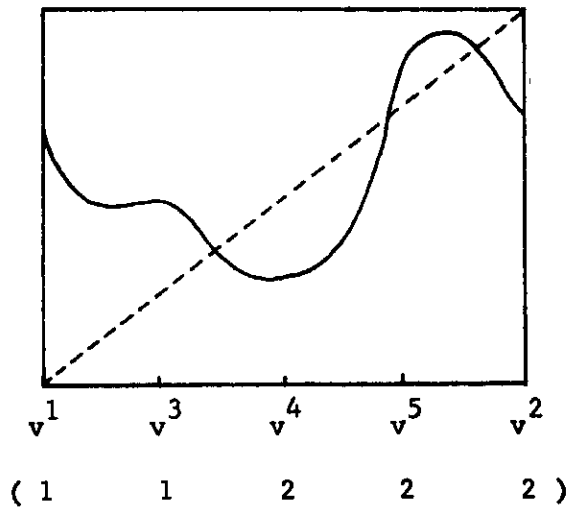

FIGURE 6

$f(v^j) < v^j$. The combinatorial lemma, which is trivial for this one dimensional problem, is simply that there must exist one small interval, in this case (v^3, v^4) whose two end points are differently labeled. If the grid were very fine, this combinatorial lemma would permit us to select two points, very close together, for which $f(x) - x$ has opposite signs. Any point in the interval will serve as an approximate fixed point of the mapping, with the degree of approximation (the closeness of x to $f(x)$) controlled by the fineness of the grid, and the modulus of continuity of f . Of course, a proof of the existence of a true rather than approximate fixed point involves continued refinement of the grid, and the selection of a convergent subsequence of approximants.

Our first generalization of this elementary combinatorial observation will involve the special class of simplicial subdivisions specified in the following definition.

3.2. [definition] A simplicial subdivision of the simplex $S = \{x | x = (x_1, \ldots, x_n) | x_i \geq 0, \Sigma x_i = 1\}$ will be said to be <u>restricted</u> if no vertices of the subdivision, other than the n unit vectors, lie on the boundary of S.

An example of a restricted simplicial subdivision is given by the following figure.

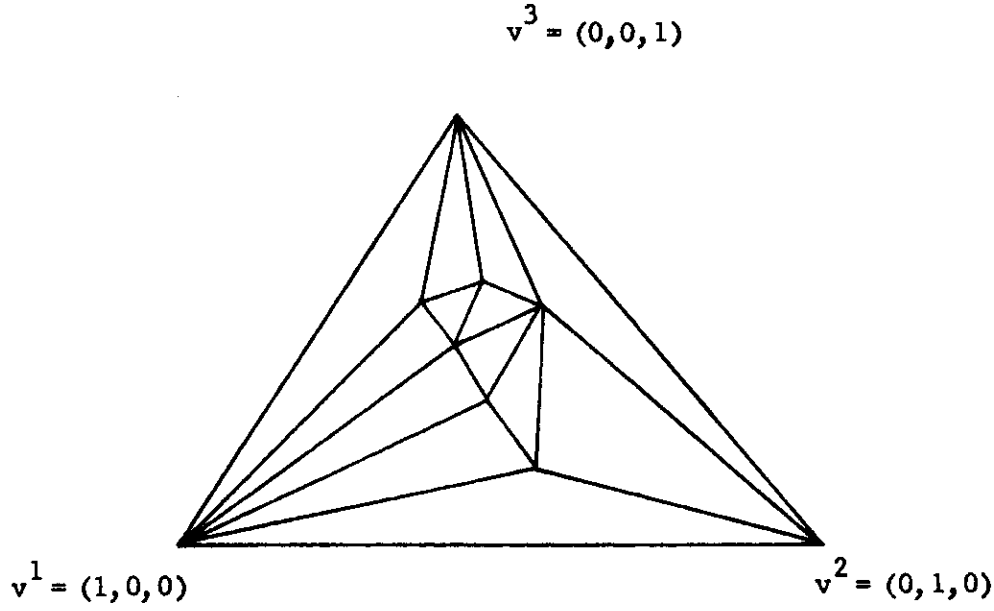

FIGURE 7

Let us consider a restricted subdivision of S, with $v^1, \ldots, v^n$ representing the first n unit vectors, and with remaining vertices $v^{n+1}, \ldots, v^k$. We shall assume that each vertex of the subdivision has associated with it an integer label $\ell(v^j)$ selected from the set $(1, 2, \ldots, n)$. When, in a moment, we apply our argument to Brouwer's theorem, the label associated with a given vertex will depend on the continuous mapping $f(x)$. For the present, however, the labels will be completely arbitrary aside from the condition that $\ell(v^j) = j$ for $j = 1, 2, \ldots, n$. In other words the only vertices whose labels are

prescribed in advance are those on the boundary of S.

The remarkable combinatorial lemma which lies behind Brouwer's theorem is the following

3.3. [lemma] Let the simplicial subdivision be restricted, and let the integer labels $\ell(v^j)$ be arbitrary members of the set $(1, 2, \ldots, n)$ aside from the condition that $\ell(v^j) = j$ for $j = 1, 2, \ldots, n$. Then there exists at least one simplex in the simplicial subdivision all of whose labels are distinct.

Avoiding for the moment the question of why one would want to find a simplex with distinct labels, we shall demonstrate the existence of such a simplex by an explicit computational procedure based on an argument which first appears in Lemke and Howson [38] and Lemke [37].

We begin by considering that unique simplex in the subdivision whose vertices are the $n-1$ unit vectors $v^2, v^3, \ldots, v^n$ and a single other vertex, say v^j. The vertices $v^2, v^3, \ldots, v^n$ will have the labels $2, 3, \ldots, n$. If the vertex v^j is labeled 1 the algorithm terminates immediately with a completely labeled simplex. Otherwise $\ell(v^j)$ will be one of the integers $2, 3, \ldots, n$. We proceed by eliminating that vertex whose label agrees with $\ell(v^j)$, arriving at a new simplex in the subdivision. Again we determine the label associated with the new vertex which has just been introduced. If this label is 1, we terminate; otherwise the process continues by eliminating that old vertex in the subdivision whose label agrees with that of the vertex just introduced.

At each stage of the algorithm we will be faced with a simplex in the subdivision whose n vertices bear the $n-1$ labels $2, 3, \ldots, n$.

A pair of the vertices, one of which has just been introduced, will have a common label. We continue by removing that member of this pair which has not just been introduced.

Figure 8 illustrates a typical example of a path of simplices followed by the algorithm.

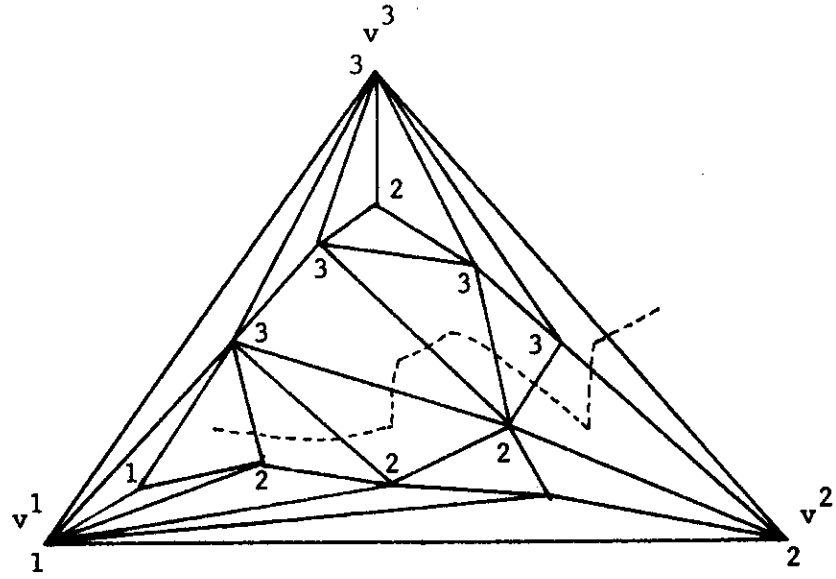

FIGURE 8

There are a finite number of simplices in the subdivision. We shall demonstrate that the algorithm never returns to a simplex which it previously encounters. Assuming this to be correct the algorithm must terminate after a finite number of iterations. But termination can only occur if we have reached a simplex all of whose labels are different, or if we arrive at a simplex $n-1$ of whose vertices lie on the boundary of S, with the remaining vertex about to be removed. Such a boundary simplex would, however, have to contain the vertices $v^2, \ldots, v^n$, since if the vertex v^1 appears in a simplex encountered in the algorithm the label 1 would have been obtained, and the algorithm would have terminated.

We see, therefore, that a proof that the algorithm terminates in a finite number of iterations, with a simplex whose labels are distinct, consists in showing that the algorithm never returns to the same simplex. Consider the <u>first</u> simplex which is revisited. If it is not the initial simplex it can be arrived at in one of two possible ways--through one or another of the adjacent simplices with n-1 distinct labels. But both

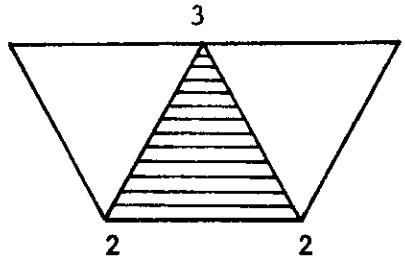

FIGURE 9

of these adjacent simplices would have been encountered during the first visit and our simplex is therefore not the first simplex to be revisited. A similar argument demonstrates that the initial simplex (with vertices $v^2, v^3, \ldots, v^n$ and one other vertex) is not the first simplex to be revisited. This concludes the proof of Lemma 3.3.

The customary proofs of Brouwer's theorem make use of a combinatorial lemma known as Sperner's lemma which is a generalization of 3.3 to arbitrary simplicial subdivisions of the simplex. Consider now a simplicial subdivision many of whose vertices lie on the boundary of S. The vertices v^j will again be labeled with integer labels $\ell(v^j)$ selected from the set $(1, 2, \ldots, n)$. The vertices which are interior to S will have arbitrary labels. In order to guarantee the existence of a simplex all of whose labels are distinct the vertices on the

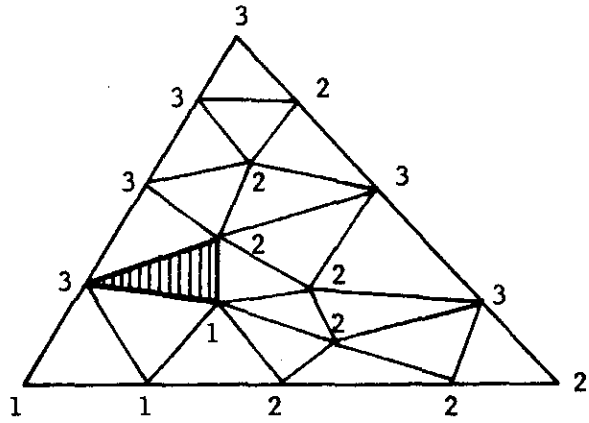

FIGURE 10

boundary faces of S will have labels which are restricted according to the following theorem.

3.4. [Sperner's lemma] Let v be an arbitrary vertex in the subdivision and assume that $\ell(v)$ is one of the indices i for which $v_i > 0$. Then there exists a simplex in the subdivision with distinct labels.

The assumption of Sperner's lemma is that a vertex on a face of S generated by a subset of the unit vectors must have a label corresponding to one of these unit vectors. The assumption is illustrated in Figure 10, in which the shaded simplex is completely labeled.

Sperner's lemma may be demonstrated from Lemma 3.3 by the following simple argument. We embed the unit simplex in a larger simplex S' with vertices $s^1, \ldots, s^n$. We then extend the simplicial subdivision of S to a <u>restricted</u> simplicial subdivision of S' by introducing a number of simplices obtained in the following way: take an arbitrary

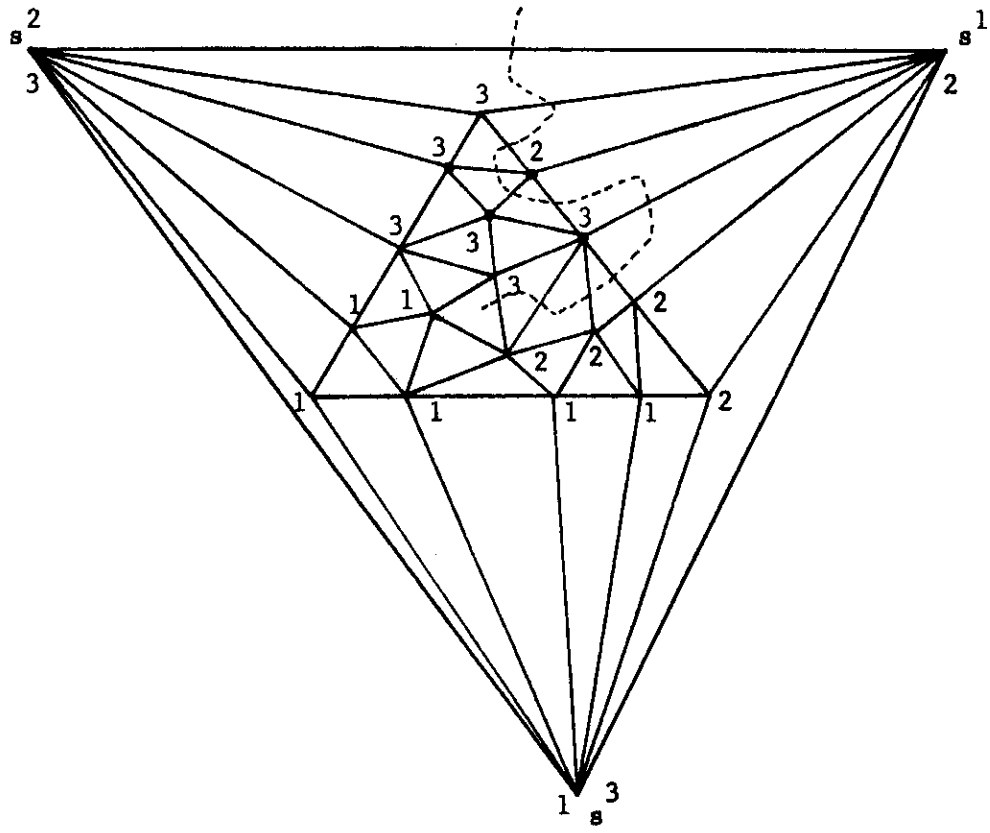

FIGURE 11

subset T of (1, 2, ..., n) with $t < n$ members. Consider a collection of n-t vertices on the face of S defined by $x_i = 0$ for $i \in T$, and which lie in a single simplex of the subdivision of S. These n-t vertices in S are augmented by the t vertices s^i for $i \in T$ in order to define a simplex in the larger subdivision.

Lemma 3.3 will be applied to the subdivision of S'. We must associate a distinct label with each of the new vertices $s^1, \ldots, s^n$ and we wish to do this in such a fashion that the completely labeled simplex obtained by applying 3.3 will contain none of the new vertices.

Let us define $\ell(s^i)$ to be equal to i+1 modulo n. In other

words $\ell(s^1) = 2$, $\ell(s^2) = 3$, ..., $\ell(s^n) = 1$. A completely labeled simplex may then contain the vertices s^i for i in T, and $n-t$ other vertices on the face $x_i = 0$ for $i \in T$. Since these remaining vertices will, by the assumption of Sperner's lemma, have labels different from the members of T it follows that the collection of vertices s^i in the completely labeled simplex must bear all of the labels in T. This is in contradiction to $\ell(s^i) = i+1$ modulo n, unless T is the empty set and the completely labeled simplex is a member of the original simplicial subdivision of S.

Having demonstrated Sperner's lemma by means of an explicit computational procedure we turn now to Brouwer's theorem. Let us consider a continuous mapping $x \to f(x)$ of the unit simplex into itself. The mapping is specified by n functions $f_i(x)$, for $i = 1, \ldots, n$ which are defined on the unit simplex, continuous and satisfy

$$f_i(x) \geq 0, \quad \text{for } i = 1, \ldots, n, \quad \text{and}$$

$$\Sigma f_i(x) = 1.$$

A simplicial subdivision of the simplex is given, with vertices $v^1, \ldots, v^n, v^{n+1}, \ldots, v^k$. We assume that none of these vertices are fixed points of the mapping f, since otherwise Brouwer's theorem would be trivially correct. It follows that for each vertex v we must have

$$v_i > f_i(v)$$

for at least one index i. We define the label associated with the vertex v to be one of these indices. If the vertex lies on the boundary of

the simplex, with say $v_j = 0$, then the label associated with v must be different from j. The assumptions of Sperner's lemma are satisfied and the algorithm on which our proof of Sperner's lemma is based may be used to determine a completely labeled simplex.

If the simplicial subdivision has a very fine mesh, in the sense that the distance between any pair of points in the same simplex is quite small, then any point in the completely labeled simplex will be close to its image and serve as an approximate fixed point of the mapping. In order to demonstrate Brouwer's theorem completely we must consider a sequence of subdivisions whose mesh tends to zero. Each such subdivision will yield a completely labeled simplex, and as a consequence of the compactness of the unit simplex there is a convergent subsequence of completely labeled simplices all of whose vertices tend to a single point x^*. (This is, of course, the non-constructive step in demonstrating Brouwer's theorem, rather than providing an approximate fixed point.) The point x^* is the limit of points bearing all of the labels $1, 2, \ldots, n$. Since f is continuous it follows that

$$x_i^* \geq f_i(x^*) \quad \text{for all} \quad i .$$

But this can only happen if $x_i^* = f_i(x^*)$, for $i = 1, \ldots, n$. This concludes the proof of Brouwer's theorem.

Let us now show that the existence of equilibrium prices for a model of exchange follows directly from Sperner's lemma without using Brouwer's theorem as an intermediary tool. We consider the continuous excess demand functions $\xi_1(\pi), \ldots, \xi_n(\pi)$ defined on the unit price simplex

$$S = \{\pi = (\pi_1, \ldots, \pi_n) | \pi_i \geq 0, \ \Sigma \pi_i = 1\} ,$$

and satisfying the Walras law

$$\Sigma \pi_i \xi_i(\pi) \equiv 0 .$$

S is subjected to a simplicial subdivision of fine mesh, with vertices $\pi^1, \ldots, \pi^n, \pi^{n+1}, \ldots, \pi^k$. We shall label a vertex π of this subdivision with an integer i for which $\xi_i \leq 0$.

In order to satisfy the hypothesis of Sperner's lemma we must show, that for any price vector π, there is at least one coordinate with $\pi_i > 0$ and $\xi_i(\pi) \leq 0$; this will provide us with a labeling consistent with 3.4. But this is certainly correct since otherwise every coordinate i with $\pi_i > 0$ would have $\xi_i(\pi) > 0$, violating the Walras law.

We may therefore conclude that there is a simplex with distinct labels. If the subdivisions are refined we may select a subsequence of these completely labeled simplices whose vertices tend, in the limit, to a price vector π^*. Since the excess demand functions are continuous $\xi_i(\pi^*) \leq 0$ for all i, and π^* is indeed an equilibrium price vector.

We shall conclude this section by demonstrating an alternative combinatorial lemma, which may be viewed as a dual form of Sperner's lemma. As before we consider a simplicial subdivision of the simplex S many of whose vertices lie on the boundary. The vertices v^j which are interior to the simplex will be given an arbitrary integer label $\ell(v^j)$ selected from the set $(1, 2, \ldots, n)$. As distinct from Sperner's lemma, however, a vertex on the boundary will be given an integer label associated with one of its coordinates which is equal to zero.

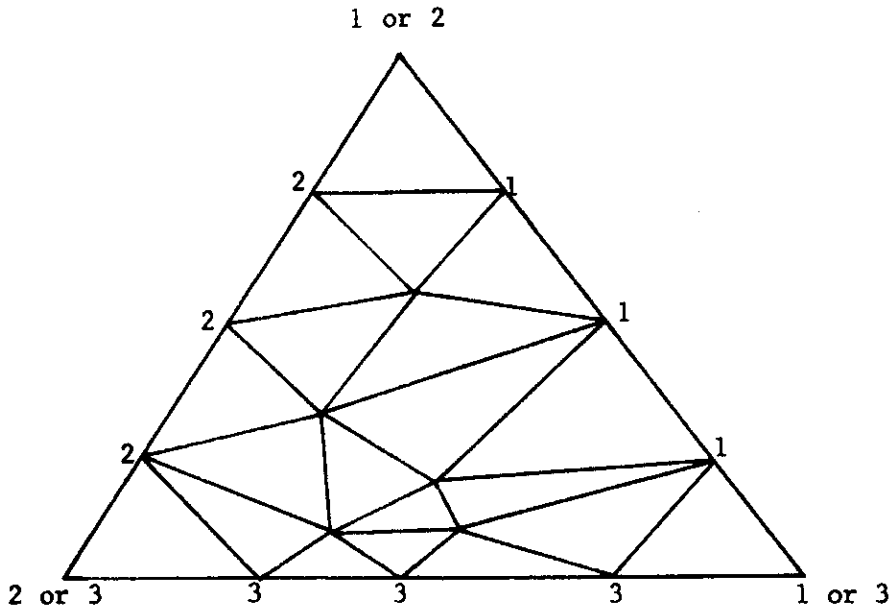

FIGURE 12

In order to guarantee the existence of a completely labeled simplex, the subdivision must be sufficiently fine. For example if there is only one simplex in the subdivision, the simplex S, itself, then the vertices can clearly be labeled, consistently with the above rule, and with some label omitted.

3.5. [lemma] Assume that no simplex in the subdivision has a non-empty intersection with every face $x_i = 0$. Let $\ell(v)$ be an arbitrary member of the set $(1, 2, \ldots, n)$ if v is a vertex which is interior to S. If v is on the boundary then $\ell(v)$ is equal to one of the indices i for which $v_i = 0$. Then there exists at least one simplex all of whose labels are distinct.

As in the proof of Sperner's lemma, we embed S in a larger simplex S' with vertices $s^1, \ldots, s^n$. The simplicial subdivision is extended to a restricted subdivision as before, i.e. we consider $n-t$ vertices on the face defined by $x_i = 0$ for i in some index set T (with $t = |T|$),

and which lie in a single simplex of the original subdivision. We then augment these vertices by the t vertices s^i for $i \in T$.

The vertices $s^1, \ldots, s^n$ will be labeled according to the rule $\ell(s^i) = i$, differing from the rule used in the proof of Sperner's lemma. Lemma 3.3 is then applied to this restricted subdivision so as to produce a completely labeled simplex. In order to demonstrate 3.5 we must show that none of the vertices $s^1, \ldots, s^n$ are contained in the completely labelled simplex.

Assume, to the contrary that the simplex contains s^i for $i \in T$, and $n-t$ vertices of a simplex in the original subdivision lying on the face $x_i = 0$ for $i \in T$. Since s^i bears the label i, the $n-t$ vertices must bear all of the labels in T^c. But these vertices all lie on the boundary of S, and their reason for bearing a given label must be that the corresponding coordinate is equal to 0. It follows that the convex hull of these $n-t$ vertices has a non-empty intersection with each face $x_i = 0$, for $i = 1, 2, \ldots, n$. This contradicts the assumption of 3.5.

Lemma 3.5 may be used as a substitute for Sperner's lemma in proving Brouwer's theorem, or in demonstrating the existence of equilibrium prices in a model of exchange. In the latter problem a price vector π which is a vertex of the subdivision interior to S is given a label i for which $\xi_i(\pi)$ is maximal. The price vectors on the boundary receive a label corresponding to a zero coordinate. A completely labeled simplex is then found for each of a sequence of subdivisions whose mesh tends to zero. Consider a subsequence of completely labeled simplices all of whose vertices tend to the common price vector $\pi^* = (\pi_1^*, \ldots, \pi_n^*)$. For any coordinate i with $\pi_i^* > 0$ we must have $\xi_i(\pi^*) \geq \xi_j(\pi^*)$ for

all j. It follows that $\xi_i(\pi^*)$ are all equal to some common value c for those i with $\pi_i^* > 0$, and that $\xi_i(\pi^*) \leq c$ if $\pi_i^* = 0$. But from the Walras law,

$$0 = \Sigma \pi_i^* \xi_i(\pi^*) = c ,$$

and π^* is indeed an equilibrium price vector.

IV. THE NON-RETRACTION THEOREM

Brouwer's theorem is frequently demonstrated by an appeal to a well known topological theorem which asserts that there is no continuous mapping of the unit simplex which carries all points in the simplex into the boundary, and which maps each boundary point into itself. It will be instructive for us to see the relationship between these two theorems and to interpret our combinatorial lemma 3.3 as an example of the non-retraction theorem.

To see that Brouwer's theorem follows from the non-retraction theorem let $x \to f(x)$ be a continuous mapping of the unit simplex into itself. If the mapping has no fixed points we can define a continuous retraction to the boundary of S in the following way.

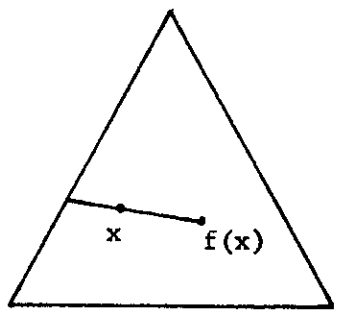

FIGURE 13

For each x draw the straight line originating at the image $f(x)$ and passing through x. The straight line will intersect the boundary of the simplex at a point which we call $g(x)$. Clearly $g(x)$ is a continuous function of x, and for any x on the boundary of the simplex $g(x) \equiv x$; g is therefore a continuous retraction of x onto the boundary. Since this cannot be, the original mapping must have a fixed point.

Conversely Brouwer's theorem may be used to demonstrate the non existence of a continuous retraction g. We simply compose g with a continuous mapping of the boundary of the simplex onto itself which has no fixed points. (For example the mapping $(x_1, x_2, \ldots, x_n) \rightarrow (x_n, x_1, \ldots, x_{n-1})$.) The composite mapping will have no fixed points and therefore contradict Brouwer's theorem.

Now let us return to the situation described in lemma 3.3. The unit simplex is given a <u>restricted</u> simplicial subdivision with vertices $v^1, \ldots, v^n, v^{n+1}, \ldots, v^k$. All of these vertices, aside from the first n, are assumed to be strictly interior to the simplex. In addition each vertex is given an integer label $\ell(v)$, which is selected without restriction from the set $(1, 2, \ldots, n)$, aside from the requirement that $\ell(v^j) = j$ for $j = 1, \ldots, n$.

These integer labels may be used to define a continuous piece-wise linear mapping $g(x)$ of the unit simplex into itself. For each vertex v in the subdivision we define $g(v) = v^{\ell(v)}$, in other words to be that one of the first n vertices whose label agrees with that of v. We then extend the mapping linearly throughout each simplex in the subdivision. If a simplex in the subdivision has vertices $v^{j_1}, v^{j_2}, \ldots, v^{j_n}$ then a point in the simplex has the form

$$x = \alpha_1 v^{j_1} + \alpha_2 v^{j_2} + \ldots + \alpha_n v^{j_n} ,$$

with $\alpha_i \geq 0$, $\Sigma \alpha_i = 1$. We then define

$$g(x) = \alpha_1 g(v^{j_1}) + \alpha_2 g(v^{j_2}) + \ldots + \alpha_n g(v^{j_n}) .$$

The mapping g is clearly continuous, and as a consequence of the assumption that the subdivision is restricted, $g(x) \equiv x$ for every boundary point of the simplex. This follows from the observation that a boundary point of S is a convex combination of a subset of the first n vectors, each of which is mapped into itself by g .

If Lemma 3.3 were false, no simplex in the subdivision would have a full set of labels. But the n vertices of a simplex for which, say, the label 1 is missing would all be mapped, under g , into that face of the boundary of S given by $x_1 = 0$. Every point on such a simplex would also be mapped, by our construction, into the same face. We see therefore that the absence of a simplex with a complete set of labels permits us to construct a continuous piece-wise linear retraction of the simplex into itself.

The constructive argument given for Lemma 3.3 may also be interpreted in terms of $g(x)$, as first suggested by Hirsch [27]. Let c be a vector on the boundary of S , whose first coordinate is zero, and whose remaining coordinates are strictly positive. We shall demonstrate that there exists a continuous path $x(t)$ which is linear in each simplex of the subdivision of the simplex through which it passes and which satisfies $g(x(t)) \equiv c$. Moreover the path will have one end point at c , the other end point in a completely labeled simplex and trace out

precisely that sequence of simplices which appear in the proof of 3.3.

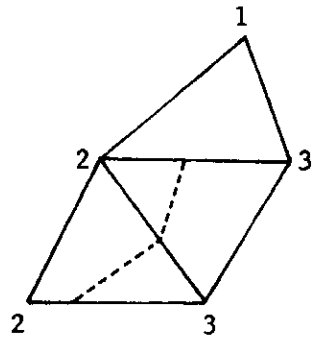

FIGURE 14

The simplices arising in 3.3 have the property that their n vertices bear the $n-1$ labels $2, 3, \ldots, n$. One of these labels will appear on two distinct vertices. Two faces of the simplex can be constructed by dropping either one of these two vertices; the $n-1$ vertices on each of these faces will have a full set of labels $2, 3, \ldots, n$. It follows that there is an interior point on each of these two faces for which $g(x) = c$. Since the mapping is linear the straight line segment connecting these two points will also satisfy $g(x) = c$.

This continuous piece-wise linear path starts at the boundary point c and passes through a sequence of simplices bearing the $n-1$ labels $2, 3, \ldots, n$. It terminates upon reaching a completely labeled simplex since such a simplex would have no other face on which $g(x) = c$ has a solution. This does in fact provide an alternative proof for Lemma 3.3 since if there were no completely labeled simplex the path could never terminate.

The relationship between fixed point theorems and the solution

of systems of equations will be explored later in this chapter.

V. A SPECIFIC SUBDIVISION

The algorithm described in Section III approximates a fixed point of a continuous mapping by moving systematically through a sequence of simplices in the simplicial subdivision. At each iteration we are given a particular simplex with vertices, say, $v^{j_1}, \ldots, v^{j_n}$. Depending on the particular assignment of integer labels a specific vertex in the simplex is removed; we move to that unique simplex in the subdivision which has $n-1$ vertices in common with the remaining vertices of the original simplex. If the method is to be implemented effectively on a computer the simplicial subdivision must be such that this replacement operation is easy to carry out. Moreover the vertices of the simplicial subdivision must be scattered with some regularity throughout the unit simplex so that the degree of approximation is independent of the region of the simplex in which the approximate fixed point happens to lie. The particular subdivision, which was brought to the attention of researchers in this field by Hansen [23] and Kuhn [35] solves these two problems admirably.

We begin our discussion by describing this particular simplicial subdivision for n dimensional Euclidean space. The n unit vectors will be denoted by $e^1, e^2, \ldots, e^n$. With this notation a simplex will have $n+1$ vertices $v^0, v^1, \ldots, v^n$, which will consist of the points in R^n with integral coordinates.

5.1. [definition] A typical simplex in the subdivision will be defined by an integral point b (called the base point) and a permutation $(\varphi_1, \ldots, \varphi_n)$ of the integers $1, 2, \ldots, n$. The vertices are then given by

$$v^0 = b$$

$$v^1 = v^0 + e^{\varphi_1}$$

$$v^2 = v^1 + e^{\varphi_2}$$

$$\vdots$$

$$v^n = v^{n-1} + e^{\varphi_n} .$$

The following figure illustrates this simplicial subdivision for the plane. The shaded simplex has the base point $b = (1, 0)$ and the

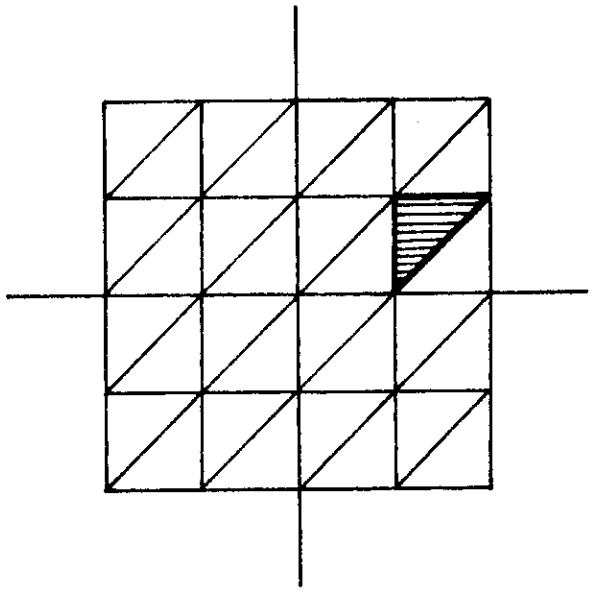

FIGURE 15

permutation $(\varphi_1, \varphi_2) = (2,1)$. Its vertices consist therefore of

$$v^0 = b \qquad = (1,0)$$

$$v^1 = v^0 + e^2 = (1,1)$$

$$v^2 = v^1 + e^1 = (2,1) \ .$$

There are two distinct types of simplices in the plane: one for each of the two permutations of (1,2). In general there will be $n!$ distinct simplices with every simplex in the subdivision equivalent under translation to one of them. Figure 16 illustrates the six distinct simplices for $n = 3$.

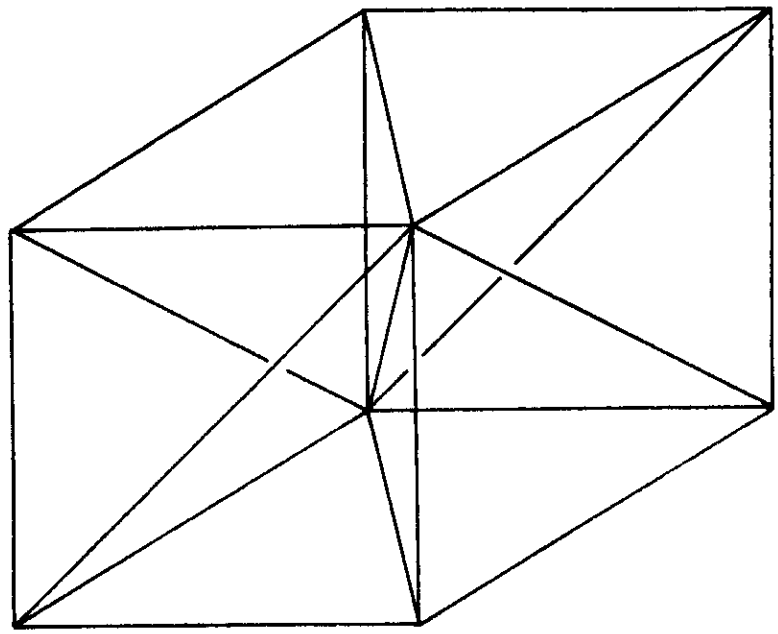

FIGURE 16

Let us consider a simplex with vertices

$$
\begin{aligned}
v^0 &= b \\
v^1 &= v^0 + e^{\varphi_1} \\
&\vdots \\
v^j &= v^{j-1} + e^{\varphi_j} \\
v^{j+1} &= v^j + e^{\varphi_{j+1}} \\
&\vdots \\
v^n &= v^{n-1} + e^{\varphi_n} ,
\end{aligned}
$$

and inquire about the replacement for a given vertex v^j ; for the moment we assume that $0 < j < n$. The new simplex will contain the vertices $v^0, v^1, \ldots, v^{j-1}$. In order to bring this about it is sufficient to have the same base point b and to have the new permutation $\hat{\varphi}_1, \hat{\varphi}_2, \ldots, \hat{\varphi}_n$ agree with the old as far as its first $j-1$ members are concerned, i.e. $\hat{\varphi}_1 = \varphi_1, \ldots, \hat{\varphi}_{j-1} = \varphi_{j-1}$.

In order not to have v^j in the new simplex we must have $\hat{\varphi}_j \neq \varphi_j$. But $v^{j+1} = v^{j-1} + e^{\varphi_j} + e^{\varphi_{j+1}}$. Since v^{j+1} must remain in the new simplex it follows that $(\hat{\varphi}_j, \hat{\varphi}_{j+1}) = (\varphi_{j+1}, \varphi_j)$. In other words the adjacent simplex with the same vertices, other than v^j , is found by taking the same base point b and the permutation in which φ_j and φ_{j+1} are transposed. The replacement $\hat{v}^j$ is then given by

$$
\begin{aligned}
\hat{v}^j &= v^{j-1} + e^{\varphi_{j+1}} \\
&= v^{j-1} + v^{j+1} - v^j .
\end{aligned}
$$

This is a remarkably simple description of the replacement which can be programmed for the computer with great ease.

If v^0 is being replaced, then we consider the new base point $\hat{b} = v^1$, and the permutation

$$(\hat{\varphi}_1, \hat{\varphi}_2, \ldots, \hat{\varphi}_n) = (\varphi_2, \ldots, \varphi_n, \varphi_1)$$

resulting in the vertices

$$\hat{v}^0 = \hat{b} = v^1$$

$$\hat{v}^1 = \hat{v}^0 + e^{\hat{\varphi}_1} = v^1 + e^{\varphi_2} = v^2$$

$$\vdots$$

$$\hat{v}^{n-1} = \hat{v}^{n-2} + e^{\hat{\varphi}_{n-1}} = v^{n-1} + e^{\varphi_n} = v^n, \quad \text{and}$$

$$\hat{v}^n = \hat{v}^{n-1} + e^{\hat{\varphi}_n} = v^n + e^{\varphi_1}.$$

We see that in this case the replacement for v^0 is given by $v^n + e^{\varphi_1} = v^n + v^1 - v^0$.

A similar argument shows that the replacement for v^n is given by $v^1 + v^{n-1} - v^n$. These rules may be summarized by the following theorem.

5.2. [theorem] Let the vertices of a simplex in the simplicial subdivision be given by $v^0, v^1, \ldots, v^n$, according to 5.1. The replacement for an arbitrary vertex v^j is given by

$$v^{j-1} + v^{j+1} - v^j,$$

with the superscripts interpreted modulo n.

In practice we store the vertices of the current simplex as columns in a matrix. When a given column is removed it is replaced by the sum of its two adjacent columns minus itself, with the interpretation that

columns 0 and n are adjacent. The following examples illustrate this remark.

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 2 & 2 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix}.$$

From the point of view of numerical computation, nothing more than this explicit rule for the replacement operation is really required. To be complete, however, it may be useful to present the details of an argument which verifies that we have indeed constructed a simplicial subdivision of R^n. We begin by showing that an arbitrary vector $x = (x_1, \ldots, x_n)$ is contained in at least one simplex of the subdivision.

5.3. [theorem] A necessary and sufficient condition that x be contained in the simplex defined by the base 0 and the permutation $\varphi_1, \ldots, \varphi_n$ is that

$$1 \geq x_{\varphi_1} \geq x_{\varphi_2} \geq \ldots \geq x_{\varphi_n} \geq 0 .$$

If x is a convex combination of the vertices $v^0, v^1, \ldots, v^n$ defined by 5.1, with $b = 0$, then

$$x = \sum_{j=0}^{n} \alpha_j \left(\sum_{\ell=1}^{j} e^{\varphi_\ell} \right)$$

$$= \sum_{\ell=1}^{n} e^{\varphi_\ell} \left(\sum_{j=\ell}^{n} \alpha_j \right) ,$$

with $\alpha_j \geq 0$, $\sum_{0}^{n} \alpha_j = 1$. It follows that

$$x_{\varphi_\ell} = \sum_{j=\ell}^{n} \alpha_j ,$$

and therefore

$$x_{\varphi_\ell} - x_{\varphi_{\ell+1}} = \alpha_\ell \geq 0 , \quad \text{for} \quad \ell = 1, \ldots, n-1 .$$

Also $x_{\varphi_n} = \alpha_n \geq 0$ and in addition $1 \geq \sum_{\ell=1}^{n} \alpha_\ell = x_{\varphi_1}$. We must therefore have

$$1 \geq x_{\varphi_1} \geq x_{\varphi_2} \geq \ldots \geq x_{\varphi_n} \geq 0 .$$

On the other hand, if this inequality is satisfied then defining

$$\alpha_\ell = x_{\varphi_\ell} - x_{\varphi_{\ell+1}} \quad \text{for} \quad \ell = 1, \ldots, n-1$$

$$\alpha_n = x_{\varphi_n} , \quad \text{and}$$

$$\alpha_0 = 1 - x_{\varphi_1}$$

will provide us with a set of weights which represents x as a convex combination of the vertices $v^0, \ldots, v^n$. This demonstrates theorem 5.3.

We see that an arbitrary vector in the unit cube will be contained in at least one simplex. By adopting a different base point a simplex can be found which contains an arbitrary vector in R^n.

Each of the $n!$ permutations $\varphi_1, \ldots, \varphi_n$ defines a simplex in the subdivision lying inside of the unit cube, to consist of those vectors with

$$1 \geq x_{\varphi_1} \geq x_{\varphi_2} \geq \ldots \geq x_{\varphi_n} \geq 0 .$$

A vector is interior to a particular simplex if all of the weights α_j are strictly positive, i.e. if all of the above inequalities are strict. It follows therefore that no two simplices in the subdivision have interior points in common.

In order to demonstrate that we have a simplicial subdivision we must show that the intersection of any two simplicies is a full face of both of them. A face of one of our simplices is obtained by setting a particular subset of the α's equal to zero and letting the remaining α's range over all non-negative values summing to 1. This is equivalent to insisting that a particular subset of the $n+1$ inequalities

$$1 \geq x_{\varphi_1} \geq x_{\varphi_2} \geq \ldots \geq x_{\varphi_n} \geq 0$$

be equalities.

The intersection of two simplices in the unit cube, one defined by the permutation $\varphi_1, \ldots, \varphi_n$ and the other by $\hat{\varphi}_1, \ldots, \hat{\varphi}_n$, will consist of those vectors in the unit cube which simultaneously satisfy the above inequalities and

$$1 \geq x_{\hat{\varphi}_1} \geq x_{\hat{\varphi}_2} \geq \cdots \geq x_{\hat{\varphi}_n} \geq 0 .$$

For example if $n = 4$, $\varphi = (4,3,2,1)$ and $\hat{\varphi} = (3,2,4,1)$ then the intersection consists of those vectors in the cube which simultaneously satisfy

$$1 \geq x_4 \geq x_3 \geq x_2 \geq x_1 \geq 0 \quad \text{and}$$

$$1 \geq x_3 \geq x_2 \geq x_4 \geq x_1 \geq 0 .$$

Together these imply

$$1 \geq x_4 = x_3 = x_2 \geq x_1 \geq 0$$

defining a full subface of each simplex. This argument, when posed in general terms, demonstrates the following theorem.

5.4. [theorem] The collection of simplices defined by 5.1 is a simplicial subdivision of R^n.

In order to implement our constructive approach to fixed point theorems, we require a simplicial subdivision of the unit simplex,

$$S = \{x = (x_0, \ldots, x_n) \mid x_i \geq 0, \sum_0^n x_i = 1\} ,$$

and with two vectors in the same simplex close to each other. We shall construct such a simplicial subdivision with vertices $(k_0/D, k_1/D, \ldots, k_n/D)$ where D is a large positive integer and $k_0, \ldots, k_n$ non-negative integers summing to D. To do this we simply modify the definition given in 5.1 by letting the base point b be an integral vector in R^{n+1} whose coordinates sum to D, and by redefining $e^1, \ldots, e^n$ to be the n vectors in R^{n+1} given by

$$e^1 = (1, -1, 0, \ldots, 0)$$

$$e^2 = (0, 1, -1, \ldots, 0)$$

$$\vdots$$

$$e^n = (0, 0, 0, \ldots, 1, -1) .$$

The simplices defined by

$$k^0 = b$$

$$k^1 = k^0 + e^{\varphi_1}$$

$$\vdots$$

$$k^n = k^{n-1} + e^{\varphi_n}$$

will have all of their vertices on the plane $\sum_0^n k_j = D$, and will form a simplicial subdivision of this plane since they are obtained from our previous subdivision of R^n by a linear transformation.

We notice that the i^{th} coordinate of any two vertices of the same simplex are either equal or differ by 1. This implies that none of the coordinate hyperplanes $x_i = 0$ intersect any of the simplices in an interior point; we can therefore select those simplices which are unambiguously in the non-negative orthant. After division by D this

provides us with a simplicial subdivision of the unit simplex with the property that two vectors in the same simplex x and x' satisfy $|x_i - x_i'| \leq 1/D$ for all i.

It should be noticed that the replacement operation described in 5.2 holds, in precisely the same form, for this simplicial subdivision. A sequence of replacements is illustrated in the following figure.

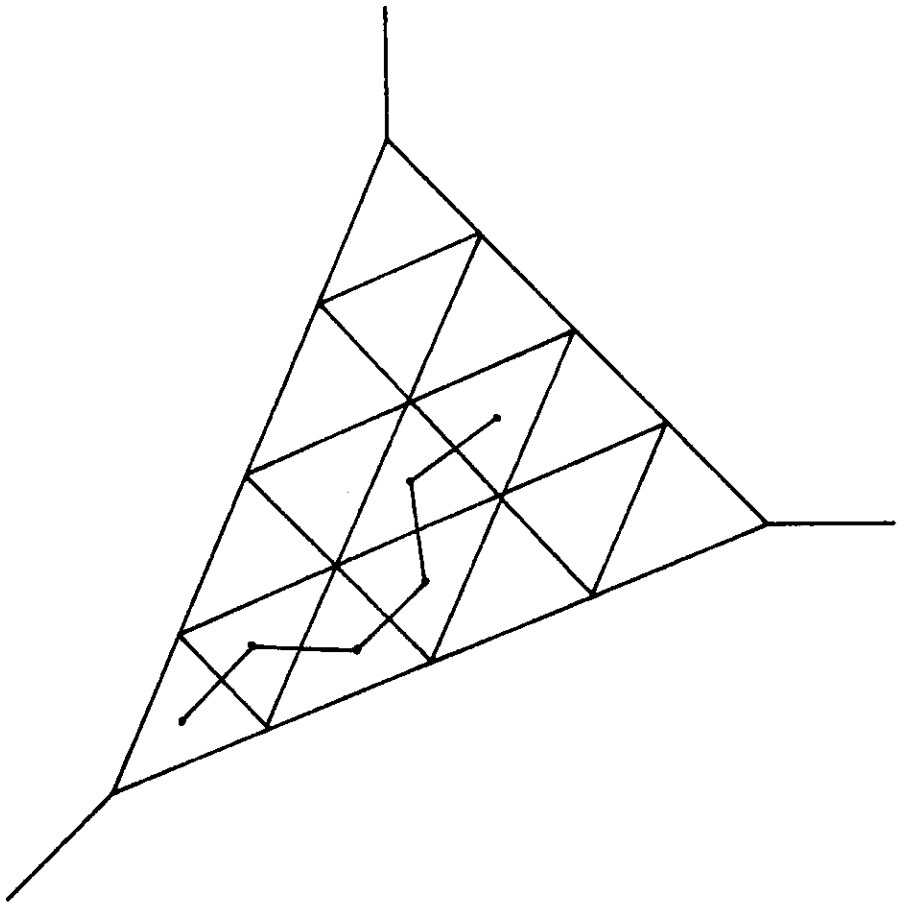

FIGURE 17

In this figure $D = 4$; the numerators of the vertices in the sequence of simplices are as follows

$$\begin{bmatrix} 4 & 3 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 2 & 3 & 3 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 2 & 2 & 3 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 2 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

VI. A NUMERICAL EXAMPLE

In this section we shall give an example of a general equilibrium model which we shall solve by the use of Lemmas 3.4 and 3.5 and the particular simplicial subdivision of the last section. Our previous discussion of Sperner's lemma involved embedding the unit simplex in a larger simplex and extending the simplicial subdivision to a restricted subdivision of the larger simplex (see Figure 11). Since the new subdivision will not be of the type described in the previous section, the very elementary replacement operation cannot be employed without some modification. Before presenting the numerical example I shall describe Kuhn's method [35] for initiating the algorithm, which avoids this difficulty and, in addition, permits us to start our calculation at any boundary point of the simplex.

Let the unit simplex $S = \{x = (x_1, \ldots, x_n) | x_i \geq 0, \sum_{i=1}^{n} x_i = 1\}$ be given a simplicial subdivision of the type described in the previous section. We select a large positive integer D and consider as vertices in the subdivision the points $(k_1/D, \ldots, k_n/D)$ with $k_1, \ldots, k_n$ non-negative integers summing to D. The numerators of the vertices of any particular simplex in the subdivision can be written as the columns in an $n \times n$ matrix

$$K = \begin{bmatrix} k_1^1 & \cdots & k_1^n \\ \vdots & & \vdots \\ k_n^1 & & k_n^n \end{bmatrix}$$

The arguments of the previous section imply that the columns of K can be

ordered in such a way that $k^j = k^{j-1} + e^{\varphi_j}$, where e^i is a vector whose $(i-1)^{st}$ coordinate is $+1$, whose i^{th} coordinate is -1 and whose remaining coordinates are 0. Moreover $\varphi_2, \ldots, \varphi_n$ is a permutation of the set $(2, \ldots, n)$. In other words each column is identical with its predecessor (interpreted modulo n) except for the entries in two successive rows, the first of which is a single unit higher and the second a single unit lower than the entries in the preceeding column.

The vertices k^j/D are given an integer label $\ell(k^j)$ taken from the set $(1, 2, \ldots, n)$ and subject to the requirement that a vector receive a label corresponding to one of its positive coordinates. Let us attempt to initiate our search for a completely labeled simplex at a point $(0, k_2^*, \ldots, k_n^*)$ on that face of the simplex whose 1^{st} coordinate is 0.

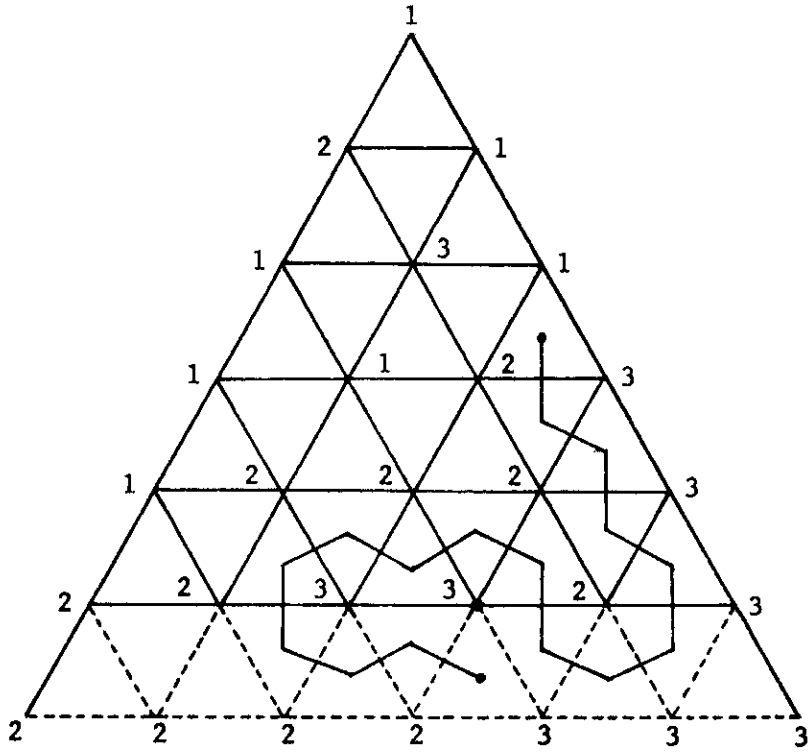

FIGURE 18

We begin by extending the simplex to include those vertices whose first coordinate is given by $k_1 = -1$. These additional vertices will be given an integer label, from the set $(2, \ldots, n)$, in such a way that there is a <u>unique</u> simplex in the extended subdivision, which consists of a single vertex with $k_1 = 0$, $(n-1)$ vertices with $k_1 = -1$, and with the latter set of vertices bearing all of the labels $2, \ldots, n$.

6.1. [labeling rule] A vertex given by $(-1, k_2, \ldots, k_n)$, with $k_i \geq 0$, for $i = 2, \ldots, n$ and $\sum_2^n k_j = D+1$, will be labeled with the first integer for which $k_i > k_i^*$.

The columns of the matrix

$$\begin{bmatrix} -1 & \cdots & -1 & -1 & 0 \\ k_2^* & & k_2^* & k_2^*+1 & k_2^* \\ k_3^* & & k_3^*+1 & k_3^* & k_3^* \\ \vdots & & \vdots & \vdots & \vdots \\ k_n^*+1 & & k_n^* & k_n^* & k_n^* \end{bmatrix}$$

represent a simplex in this extended subdivision. The first $n-1$ vectors bear the labels $n, n-1, \ldots, 2$; the final vector bears a label different from 1 . The algorithm can be initiated at this simplex and can only run into difficulty if it encounters another simplex

$$\begin{bmatrix} -1 & \cdots & -1 & 0 \\ k_2^1 & & k_2^{n-1} & k_2^n \\ \vdots & & \vdots & \vdots \\ k_n^1 & & k_n^{n-1} & k_n^n \end{bmatrix}$$

whose first $n-1$ columns bear all of the labels $n, \ldots, 2$, say in that particular order. For this to be true, however, we must have

$$k^j_{n-j+1} > k^*_{n-j+1} \quad \text{for} \quad j = 1, \ldots, n-1 .$$

Since no two entries in the same row of this matrix differ by more than unity

$$k^j_i \geq k^*_i \quad \text{for} \quad j = 1, \ldots, n-1 \quad \text{and} \quad i = 2, \ldots, n .$$

But $\sum_{i=2}^{n} k^j_i = \sum_{i=2}^{n} k^*_i + 1$, and it follows that the second matrix is identical to the one initiating the algorithm.

Let us now consider an example of an exchange economy involving 5 types of consumers and 10 commodities. The i^{th} consumer will own, prior to trade, a specific vector w^i, as given by the following table:

Initial Stock of Commodities

Consumer										
1	0.6	0.2	0.2	20.0	0.1	2.0	9.0	5.0	5.0	15.0
2	0.2	11.0	12.0	13.0	14.0	15.0	16.0	5.0	5.0	9.0
3	0.4	9.0	8.0	7.0	6.0	5.0	4.0	5.0	7.0	12.0
4	1.0	5.0	5.0	5.0	5.0	5.0	5.0	8.0	3.0	17.0
5	8.0	1.0	22.0	10.0	0.3	0.9	5.1	0.1	6.2	11.0

The i^{th} consumer's demand functions will be derived from a utility function

$$u(x) = \sum_{1}^{10} a_i^{1/b_i} x_i^{1 - 1/b_i} .$$

The following table describes the parameters a_j for each consumer.

Utility Parameters

Consumer										
1	1.0	1.0	3.0	0.1	0.1	1.2	2.0	1.0	1.0	.07
2	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0
3	9.9	0.1	5.0	0.2	6.0	0.2	8.0	1.0	1.0	0.2
4	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
5	1.0	13.0	11.0	9.0	4.0	0.9	8.0	1.0	2.0	10.0

The parameters b are given by

	b
1	2.0
2	1.3
3	3.0
4	0.2
5	0.6

The problem was solved by selecting $D = 200$, and initiating the algorithm at the center of the face $x_1 = 0$. After 6063 iterations the following matrix, whose columns are the numerators of the vertices of a completely labeled simplex, was found.

$$\begin{bmatrix}
39 & 39 & 39 & 39 & 40 & 39 & 39 & 39 & 39 & 39 \\
22 & 22 & 22 & 22 & 22 & 23 & 22 & 22 & 22 & 22 \\
19 & 19 & 19 & 19 & 19 & 19 & 20 & 19 & 19 & 19 \\
8 & 8 & 8 & 8 & 8 & 8 & 8 & 9 & 8 & 8 \\
23 & 23 & 23 & 23 & 23 & 23 & 23 & 23 & 24 & 23 \\
15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 15 & 16 \\
24 & 23 & 23 & 23 & 23 & 23 & 23 & 23 & 23 & 23 \\
21 & 22 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 \\
20 & 20 & 21 & 20 & 20 & 20 & 20 & 20 & 20 & 20 \\
9 & 9 & 9 & 10 & 9 & 9 & 9 & 9 & 9 & 9
\end{bmatrix}$$

Each of these columns, after division by 200, may be used as an approximate equilibrium price vector, yielding a vector of excess demands which is fairly close to zero. A better approximation is obtained by averaging these prices, using a variation of Newton's method, to yield

$$\pi = (.187,\ .109,\ .099,\ .043,\ .117,\ .077,\ .117,\ .102,\ .099,\ .049)\ .$$

At this price vector the excess demands are equal to zero for two decimal places.

The algorithm underlying Lemma 3.5 can also be used to solve this problem. In this latter algorithm a vertex of the subdivision k^j/D receives a label corresponding to one of its zero coordinates if it is on the boundary of the unit simplex, and otherwise a label corresponding to that commodity with the largest excess demand. To be specific we shall agree on the following convention.

6.2. [labeling rule for the second algorithm] The label associated with a vertex on the boundary of the simplex will be the <u>first</u> coordinate which is equal to zero.

Our earlier arguments for this version of the algorithm involved embedding the unit simplex in a larger simplex to which the simplicial subdivision is extended. The particular form of our simplicial subdivision, however, permits us to work directly with the unit simplex and to show that the algorithm never leads us outside of the simplex.

Let us define $d = D - n + 1$. The columns of the following matrix represents a simplex in the subdivision:

$$\begin{bmatrix} d & d+1 & d+1 & \cdots & d+1 & d \\ 1 & 0 & 1 & & 1 & 1 \\ 1 & 1 & 0 & & 1 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & 1 & & 0 & 0 \\ 0 & 0 & 0 & & 0 & 1 \end{bmatrix}$$

Labels $\quad n \quad 2 \quad 3 \quad \cdots \quad n-1 \quad n-1$

According to our labeling rule, the label 1 does not appear, all of the remaining labels do appear, and the label $n-1$ is associated with the last two columns. If the n^{th} column is removed we are carried outside of the unit simplex; we begin the algorithm, therefore, by removing column $n-1$.

The algorithm can only run into difficulty if we encounter another matrix which contains a row $(0, 0, 0, \ldots, 0, 1)$ and if the last column is to be removed because it shares a label with one of the other columns.

If, however, this row is the i^{th} row in the matrix, and $i < n$, then each of the first $n-1$ columns will receive a label $\leq i$. The label n can only appear in the last column, which, consequently, will not be removed.

We can therefore assume that the row $(0, 0, 0, \ldots, 0, 1)$ is the n^{th} row in the matrix. If the last column is to be removed, the first $n-1$ columns of the matrix will bear the labels $2, 3, \ldots, n$, and since each one of these columns is on the boundary of the simplex, the corresponding label will be borne because it is the first coordinate equal to zero in that column. It follows that each row of the matrix, other than the first, must have a zero entry, and therefore that these rows are composed entirely of zeros and ones.

The column bearing the label n must then be given by

$$\begin{bmatrix} d \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 0 \end{bmatrix},$$

i.e., the first column in our initial matrix.

The column bearing the label 2 must have at least two zero entries in it, one in the 2^{nd} row, and one in the last. Since the entries in rows $2, 3, \ldots, n$ are zero or one, if there were a third zero entry it would follow that the entry in row 1 would be $\geq d+2$. Since this is impossible we see that the column bearing the label 2 must be the second column in

our original matrix.

If this argument is continued, we see that the first $n-1$ columns of the matrix leading us outside of the simplex must be identical with those of our initial matrix. But this face can be completed to a simplex in the subdivision only by the addition of the last column of the original matrix. It follows that an exit from the unit simplex can only occur by returning to the initial simplex; this is impossible from the general arguments of Section III.

When this version of the algorithm was applied to the previous example, again with $D = 200$, termination was achieved in 2996 iterations, with the completely labeled simplex whose numerators are displayed below.

$$\begin{bmatrix}
36 & 36 & 36 & 35 & 36 & 36 & 36 & 36 & 36 & 36 \\
22 & 22 & 22 & 22 & 21 & 22 & 22 & 22 & 22 & 22 \\
20 & 20 & 20 & 20 & 20 & 19 & 20 & 20 & 20 & 20 \\
9 & 9 & 9 & 9 & 9 & 9 & 8 & 9 & 9 & 9 \\
23 & 23 & 23 & 23 & 23 & 23 & 23 & 22 & 23 & 23 \\
16 & 16 & 16 & 16 & 16 & 16 & 16 & 16 & 15 & 16 \\
24 & 24 & 24 & 24 & 24 & 24 & 24 & 24 & 24 & 23 \\
20 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 & 21 \\
20 & 19 & 20 & 20 & 20 & 20 & 20 & 20 & 20 & 20 \\
10 & 10 & 9 & 10 & 10 & 10 & 10 & 10 & 10 & 10
\end{bmatrix}$$

The more rapid termination of this version of the algorithm is entirely attributable to the fact that the vertex $(1, 0, \ldots, 0)$ is a better approximation to the true equilibrium price than the vector initiating the previous algorithm, and not to any inherent superiority of this version.

Both algorithms require a fairly substantial amount of computation in order to yield a modest accuracy. In the next several sections we shall describe a number of variations of the basic algorithm which provide a dramatic improvement in computational speed, in addition to extending the class of problems to which these methods can be applied.

VII. VECTOR LABELS

The algorithms with which we have been concerned associate with each vertex of a simplicial subdivision an integer label selected from the set $(1, 2, \ldots, n)$. When applied, for example, to determining equilibrium prices for a model of exchange the excess demands $\xi_1(\pi), \ldots, \xi_n(\pi)$ are evaluated, and the associated label corresponds, say, to that commodity with the largest excess demand.

It is clearly inefficient from a computational point of view to replace the vector of excess demands, whose evaluation may in itself be quite time consuming, by a single integer. The vector labeling procedures to be discussed in this section permit us to utilize all of the information in the excess demand vector with the expectation of greater accuracy in less computing time. In addition they are applicable to a large class of problems, including for example the approximation of a fixed point implied by Kakutani's theorem, to which the earlier methods cannot readily

be applied.

Let us consider, as before, a restricted simplicial subdivision of the unit simplex, with vertices $v^1, \ldots, v^n, v^{n+1}, \ldots, v^k$.

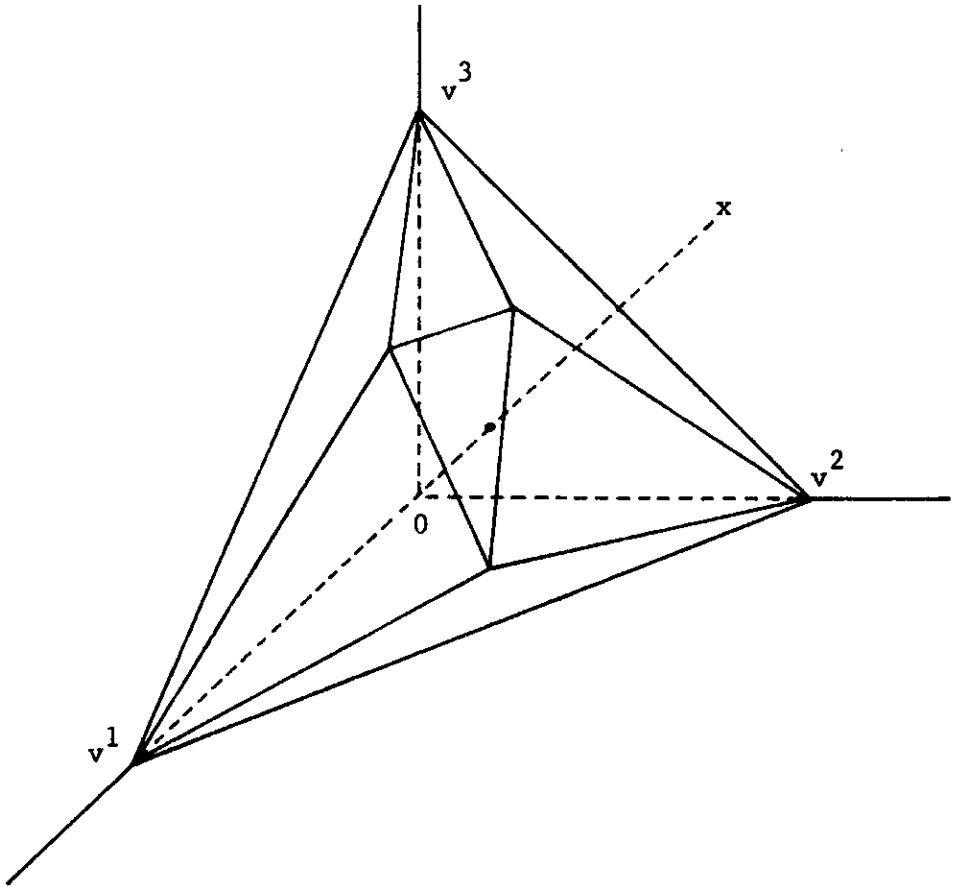

FIGURE 19

Each vertex v^j will have associated with it a label $\ell(v^j)$, which is a vector in n dimensional space. In the applications of the vector labeling algorithms these labels will depend on the particular problem being solved. For the moment, however, the labels are arbitrary, aside from the stipulation that the unit vectors defining the simplex receive the labels $\ell(v^j) = v^j$ for $j = 1, 2, \ldots, n$. We shall also be given, in advance, a specific positive vector b in R^n .

Our previous algorithms attempted to determine a simplex in the

subdivision, with vertices $v^{j_1}, v^{j_2}, \ldots, v^{j_n}$ and whose labels $\ell(v^{j_1}), \ell(v^{j_2}), \ldots, \ell(v^{j_n})$ were distinct. By a solution to the vector labeling problem we mean the determination of a simplex with vertices $v^{j_1}, \ldots, v^{j_n}$, for which the system of linear equations

$$y_{j_1} \ell(v^{j_1}) + \ldots + y_{j_n} \ell(v^{j_n}) = b ,$$

have a non-negative solution $y_{j_1}, \ldots, y_{j_n}$. The following theorem provides sufficient conditions on the labels $\ell(v^j)$ for the problem to have a solution.

7.1. [theorem] Consider a restricted subdivision of the unit simplex with vertices $v^1, \ldots, v^n, v^{n+1}, \ldots, v^k$. Each vertex v^j has associated with it a label $\ell(v^j) \in R^n$; the first n vectors have as labels $\ell(v^j) = v^j$. Assume that

$$\sum_{j=1}^{k} y_j \ell(v^j) \leq 0 ,$$

and $y_j \geq 0$ for $j = 1, \ldots, k$ implies $y_j = 0$ for all j. Then for any positive vector $b \in R^n$, there is a simplex in the subdivision with vertices $v^{j_1}, \ldots, v^{j_n}$, such that

$$\sum_i y_{j_i} \ell(v^{j_i}) = b$$

has a non-negative solution.

Theorem 7.1 has an interesting geometric interpretation which illuminates the method we shall provide for determining a correct simplex. Let us define a mapping $g(x_1, \ldots, x_n)$ from the non-negative orthant of R^n into R^n in the following way

7.2. [definition] $g(x)$ is defined to be the unique mapping satisfying

1. $g(v^j) = \ell(v^j)$ for each vertex in the subdivision,
2. $g(x)$ is linear in each simplex in the subdivision,
3. $g(x)$ is homogeneous of degree 1, i.e. $g(\lambda x) = \lambda g(x)$ for $\lambda \geq 0$.

To determine $g(x)$ for any non-negative x (different from 0) we find that λ for which λx is on the unit simplex, and therefore contained in a particular simplex with vertices $v^{j_1}, \ldots, v^{j_n}$. We then write

$$\lambda x = y_{j_1} v^{j_1} + \ldots + y_{j_n} v^{j_n},$$

with $y \geq 0$ and summing to 1, and define

$$g(\lambda x) = y_{j_1} \ell(v^{j_1}) + \ldots + y_{j_n} \ell(v^{j_n}).$$

With this interpretation we see that Theorem 7.1 is equivalent to asserting the existence of a vector x^* with $g(x^*) = b$. The validity of this assertion is a consequence of the following two properties which are immediate from the definition of $g(x)$.

7.3. [properties]

1. $g(x) = x$ for x on the boundary of the positive orthant,

2. $g(x) \leq 0$ implies $x = 0$.

The mapping $g(x)$ is piece-wise linear. The non-negative orthant of R^n is partitioned into a finite set of polyhedral cones--each one of which has n extreme rays through the origin--and such that g is linear in each cone.

Let $P = R^n_+ \times [0,1]$, the product of the non-negative orthant and the closed unit interval. The set P is naturally subdivided into a finite number of "wedges" as the following figure indicates for $n = 2$.

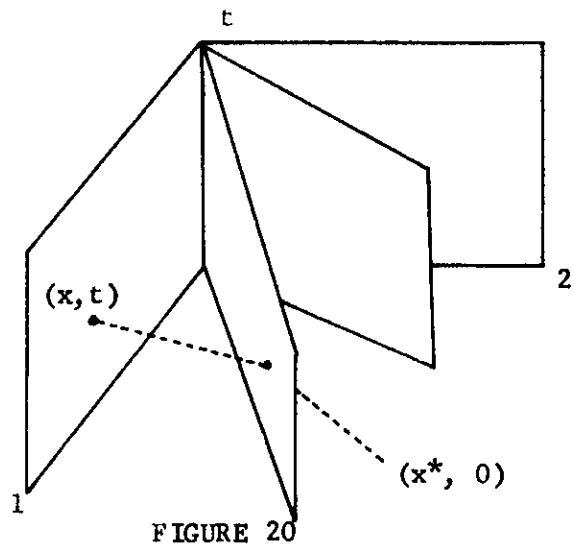

FIGURE 20

We shall extend g to a mapping G of the set P into n space in such a way that G is linear in each wedge.

7.4. [definition] Let c be a vector in R^n , strictly larger than b in each coordinate. For $(x_1, \ldots, x_n, t) \in P$ we define

$$G(x,t) = g(x) + tc \ .$$

With this definition a solution to the vector labeling problem is a vector x^* for which $G(x^*, 0) = b$. The mapping G will permit us to trace out the one dimensional family of solutions to $G(x,t) = b$ and to show that there is at least one member of the family with $t = 0$.

The boundary of the set P consists of a number of hyperplanes: the two hyperplanes given by $t = 0$ and $t = 1$, and the n hyperplanes defined by $x_i = 0$. Let us ask whether there are any solutions to $G(x,t) = b$ on the boundary of P --other than $t = 0$.

A solution on the upper boundary would require $g(x) + c = b$ and since $c > b$ this implies $g(x) < 0$, contrary to the second property in 7.3. To examine the solutions on that part of the boundary defined by $x_i = 0$, we make use of the first property in 7.3, i.e. that $g(x) = x$ on the boundary of the non-negative orthant. For such points $G(x,t) = b$ is equivalent to $x + tc = b$. These equations have a unique solution

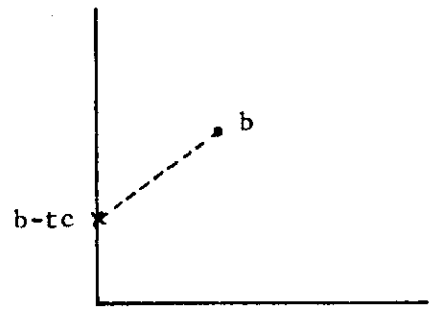

FIGURE 21

obtained by selecting that index, say i^*, for which (b_i/c_i) is minimal, setting $t = b_{i^*}/c_{i^*}$ and $x = b-ct$.

We see that there is a unique solution to the equations $G(x,t) = b$ on the boundary of P, other than that part of the boundary on which $t = 0$. The algorithm for finding x^* consists simply of following the

piecewise linear path defined by $G(x,t) = b$ from this particular boundary point until it intersects a point on the boundary with $t = 0$.

Each wedge is defined by a simplex in the subdivision with vertices $v^{j_1}, \ldots, v^{j_n}$. If

$$x = y_{j_1} v^{j_1} + \ldots + y_{j_n} v^{j_n}, \quad \text{then}$$

$$g(x) = y_{j_1} \ell(v^{j_1}) + \ldots + y_{j_n} \ell(v^{j_n}).$$

In this wedge the equations $G(x,t) = b$ therefore take the form

$$y_{j_1} \ell(v^{j_1}) + \ldots + y_{j_n} \ell(v^{j_n}) + tc = b,$$

with $y_{j_i} \geq 0$ and $0 \leq t \leq 1$. The path enters the wedge on a particular boundary face of the wedge for which one of the y variables is equal to zero and exits when another variable is equal to zero. The precise determination of these two faces can be facilitated by adopting the terminology of linear programming. Let us define a matrix L with n rows and with $k+1$ columns, where k is the number of vertices in the simplicial subdivision. The j^{th} column of L (for $j \leq k$) is the vector $\ell(v^j)$. Column $k+1$ is defined to be c.

Assume, to be specific, that the path enters the wedge at a point where $y_{j_1} = 0$. This corresponds to the statement that columns $(j_2, \ldots, j_n, k+1)$ form a feasible basis for the system of linear equations $Ly = b$. The path will exit at a point where $y_{j_1} > 0$ and some other variable is set equal to zero; but this is precisely the same as

bringing the column $\ell(v^{j_1})$ into the feasible basis and removing one of the columns in the basis by an ordinary linear programming pivot step. If the column $\ell(v^{j_i})$ is removed, the adjacent wedge is found by replacing the vertex v^{j_i} in the simplex defined by $v^{j_1}, \ldots, v^{j_n}$. If the $(k+1)^{st}$ column is removed the algorithm terminates since at this point we obtain a solution with $t = 0$.

To summarize, at each stage of the algorithm we are given a specific simplex with vertices $v^{j_1}, \ldots, v^{j_n}$. The columns of L associated with $n-1$ of these vertices and the vector c form a feasible basis for $Ly = b$. We proceed by bringing into the basis the column associated with the missing vertex, replacing the vertex whose column has been eliminated from the feasible basis and continuing. The initial configuration is the simplex $v^1, \ldots, v^n$ with v^{i*} absent and say v' included, since these $n-1$ vertices form a feasible basis along with column c. The algorithm terminates when column c is removed from the feasible basis by the pivot step.

To be completely tidy about our argument for Theorem 7.1, two observations are in order. First of all the pivot steps which are required can always be carried out. For if not the path would contain an infinite half-line in a single wedge. The variable t would necessarily be constant along this line so that $g(x)$ would be constant along a half-line extending to infinity, violating the second property of 7.3. Secondly any degeneracy in the system $Ly = b$ must be avoided by one of the devices of linear programming. Degeneracy means that the exit from a given wedge may lead to more than one adjacent wedge with a bifurcation of the piecewise linear path.

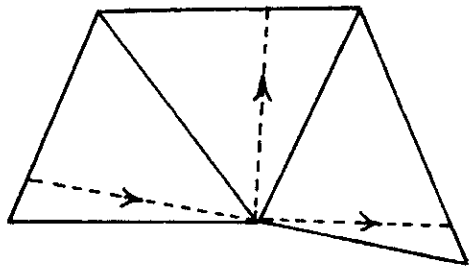

FIGURE 22

It should also be remarked that the parameter t need not be monotonic along the path, and that the set of solutions to $G(x,t) = b$ may contain piecewise linear components other than the one constructed by our algorithm. These matters are discussed in detail in [14].

In order to utilize the vector labeling algorithm it must be adapted to the regular simplicial subdivision introduced in Section V. We do this by associating with a vertex v, on the boundary of the unit simplex, the vector label $\ell(v)$ equal to the i^{th} unit vector in R^n if v_i is the <u>first</u> coordinate equal to zero. The algorithm begins, as in Section VI with the simplex, the numerators of whose vertices are given by

$$\begin{bmatrix} d & d+1 & \cdots & d+1 & d \\ 1 & 0 & & 1 & 1 \\ 1 & 1 & & 1 & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 1 & & 0 & 0 \\ 0 & 0 & & 0 & 1 \end{bmatrix}$$

Labels $\quad e^n \quad e^2 \quad \cdots \quad e^{n-1} \quad e^{n-1}$,

and whose associated vector labels are all unit vectors. The vector labels associated with the first $(n-1)$ vertices form a feasible basis, along

with the vector c, for the system $Ly = b$, if c is selected so that $b_1/c_1 \leq b_i/c_i$. The first step is to pivot into the basis the vector associated with the last column. We remove the vertex associated with column $n-1$, find the new simplex and continue. The arguments of Section VI may be used to show that we never depart from the unit simplex and terminate after a finite number of iterations with a simplex whose associated vector labels form a feasible basis for $Ly = b$.

VIII. THE GENERAL EQUILIBRIUM MODEL WITH PRODUCTION

The methods of the last section provide an exceptionally flexible technique for the numerical solution of a wide variety of problems. They can be used to approximate equilibrium prices in a general Walrasian model --with or without production--to approximate fixed points of an upper semi-continuous point to set mapping; to solve non-linear programming problems and to determine a vector in the core of an n person game. Limitations of space make it impossible to describe anything other than the application of vector labeling methods to the determination of equilibrium prices, and even in this example we shall be forced to restrict our attention to one of the many approaches which have been suggested. The reader who is curious about other applications can consult [49] or other items in the bibliography at the end of the paper.

Consider a general equilibrium model specified by a strictly positive vector w of assets prior to production, a set of market demand functions $x(\pi)$ (continuous, homogeneous of degree 1, and satisfying the Walras law $\pi \cdot x(\pi) \equiv \pi \cdot w$), and an activity analysis matrix

$$A = \begin{bmatrix} -1 & \cdots & 0 & \cdots & a_{1j} & \cdots \\ 0 & & 0 & & a_{2j} & \cdots \\ \vdots & & \vdots & & \vdots & \\ 0 & & -1 & & a_{nj} & \cdots \end{bmatrix} .$$

We make the conventional assumption that $Ay \geq 0$, $y \geq 0$, implies $y = 0$. A price vector π^* and a non-negative vector of activity levels y^* will be a solution to the general equilibrium problem if

1. $x(\pi^*) = w + Ay^*$, and
2. $\pi^* A \leq 0$.

The following assignment of vector labels to the vertices of a simplicial subdivision of the price simplex will provide an approximate solution.

8.1. [Labeling Rule]. Let π be a vertex of the simplicial subdivision. If π is on the boundary of the simplex then $\ell(\pi)$ is the i^{th} unit vector where i is the first coordinate of π equal to zero. If π is interior to the simplex we find that column a^j in the activity analysis matrix for which $\pi \cdot a^j$ is maximal. If $\pi \cdot a^j > 0$, then $\ell(\pi) = -a^j$. If $\pi \cdot a^j \leq 0$, then $\ell(\pi) = x(\pi)$. Moreover, the vector b is defined to be w.

In order to apply 7.1 we must verify that the vector labels $\ell(\pi^j)$ satisfy the condition that $\Sigma y_j \ell(\pi^j) \leq 0$, $y_j \geq 0$ implies $y_j = 0$ for all j. This can easily be shown to be a consequence of the assumption made about the activity analysis matrix plus the fact that for each price vector π, $x(\pi) \geq 0$, and is positive for at least one coordinate.

When the algorithm is applied we determine a simplex with vertices

$\pi^{j_1}, \ldots, \pi^{j_n}$. Several of these vertices will have associated vector labels which are the market demand functions evaluated at the corresponding price; the remaining vertices have labels which are the negatives of certain columns in the activity analysis matrix. With an obvious change in notation the equations $Ly = b$ can be written as

$$\Sigma\, y_j x(\pi^j) - \Sigma\, y'_\ell a^\ell \leq w \ .$$

The labeling rule 8.1 permits us to make the following statement: In the above inequality $y_j > 0$ if at prices π^j <u>all</u> activities make a profit ≤ 0 , and $y'_\ell > 0$ if for some vertex of the final simplex a^ℓ is the activity which makes the largest profit, and it is positive. Moreover the i^{th} row will be an equality unless some vertex of the final simplex has its i^{th} coordinate equal to zero.

It may be shown that the activity levels y'_ℓ and a suitable average of the prices in the final simplex are an approximate equilibrium in the sense that the two defining properties are approximately satisfied. This is a rather tedious argument, however, which can be circumvented by imagining that the problem has been solved for an infinite sequence of grids whose mesh tends to zero. We then select a subsequence of solutions, whose vertices tend to a price vector π^* and whose activity levels tend to a vector y^* . The above inequalities become

$$y^* x(\pi^*) - \Sigma\, y^*_\ell a^\ell \leq w \ .$$

The labeling rule now provides us with the following information.

8.2. [consequence of labeling rule]

1. If $y^* > 0$ then $\pi^* A \leq 0$.

2. If $y^*_\ell > 0$ then a^ℓ is a column which maximizes profit at prices π^* , and $\pi^* a^\ell \geq 0$.

3. If $\pi^*_i > 0$ then the i^{th} row in the above inequality is, in fact, an equality.

In order to show that π^* and $\{y^*_\ell\}$ form a competitive equilibrium we begin by demonstrating that $y^* > 0$.

If $y^* = 0$, we see that $-\Sigma\, y^*_\ell a^\ell \leq w$ and as a consequence of 3:

$$-\Sigma\, y^*_\ell \pi^* \cdot a^\ell = \pi^* \cdot w \ .$$

But this is impossible since the right hand side is positive and the left non-positive. We see that $y^* > 0$ and therefore $\pi^* A \leq 0$. In conjunction with 2 we can now argue that $\pi^* \cdot a^\ell = 0$ for any activity with $y^*_\ell > 0$. If follows that $\Sigma\, y^*_\ell \pi^* \cdot a^\ell = 0$, and as a consequence

$$y^* \pi^* \cdot x(\pi^*) = \pi^* \cdot w \ .$$

Making use of the Walras law, y^* is therefore equal to 1, and our inequalities become

$$x(\pi^*) \leq w + \Sigma\, y^*_\ell a^\ell \ ,$$

with equality if the i^{th} price is positive. This concludes the argument that π^* , y^* are an equilibrium solution.

In order to illustrate this algorithm we consider an example, discussed in detail in [47], of a general equilibrium model with production. There are six commodities which can be described as follows:

1. Capital available at the end of the period.
2. Capital available at the beginning of the period.
3. Skilled labor.
4. Unskilled labor.
5. Nondurable consumer goods.
6. Durable consumer goods.

The activity analysis model of production is given by the following matrix, in which the six columns representing disposal activities have been omitted.

	Activities							
Commodity	7	8	9	10	11	12	13	14
1	4	4	1.6	1.6	1.6	.9	7	8
2	-5.3	-5	-2	-2	-2	-1	-4	-5
3	-2	-1	-2	-4	-1	0	-3	-2
4	-1	-6	-3	-1	-8	0	-1	-8
5	0	0	6	8	7	0	0	0
6	4	3.5	0	0	0	0	0	0

In addition to the specification of production possibilities, we assume that there are five consumers who, at the beginning of the period own positive quantities of goods 2, 3, 4, 6 according to the following table.

	Commodity			
Consumer	2	3	4	6
1	3	5	.1	1
2	.1	.1	7	2
3	2	6	.1	1.5
4	1	.1	8	1
5	6	.1	.5	2

Each consumer will be assumed to have a C.E.S. utility function, of the type used in Section VI and with the following set of parameters.

	Utility Parameters					
Consumer						
1	4	0	0.2	0	2	3.6
2	0.4	0	0	0.6	4	1
3	2	0	0.5	0	2	1.5
4	5	0	0	0.2	5	4.5
5	3	0	0	0.2	4	2

The parameters b are given by

Consumer	b
1	1.2
2	1.6
3	.8
4	.5
5	.6

To solve this problem, a denominator $D = 200$ was selected, and the labeling rule 8.1 was used. After approximately 2200 iterations the simplex whose numerators are given by the columns of the following matrix was obtained.

$$\begin{bmatrix} 44 & 44 & 44 & 44 & 44 & 45 \\ 47 & 48 & 48 & 48 & 48 & 47 \\ 35 & 34 & 35 & 35 & 35 & 35 \\ 12 & 12 & 11 & 12 & 12 & 12 \\ 22 & 22 & 22 & 21 & 22 & 22 \\ 40 & 40 & 40 & 40 & 39 & 39 \end{bmatrix}$$

Five of these six columns have associated vector labels which are the negatives of activities 7, 9, 10, 11 and 13. The corresponding weights y_j' can be used as approximations to the equilibrium activity levels. In order to obtain an approximate equilibrium price vector the columns were averaged,

Equilibrium Prices

.220833 .238333 .174167 .059167 .109167 .198333

These prices were used to calculate the following table of profitabilities:

Activity	Level	Profit
7	0.468253	0.006000
8	0.0	-0.143333
9	3.127146	0.005833
10	0.188899	-0.005833
11	0.165613	-0.006667
12	0.0	-0.039583
13	0.365860	0.010833
14	0.0	-0.246667

and to evaluate the consumer market demand functions:

Commodity	Demand	Supply
1	11.188389	10.004686
2	0.0	1.191501
3	1.099712	2.090414
4	2.731755	3.970645
5	23.079518	21.433357
6	9.705318	9.373014

As can be seen the approximation is not exceptionally accurate; in particular the discrepancy between supply and demand is fairly large for certain commodities. An improvement in accuracy can be obtained by selecting a much finer grid but this is a very expensive procedure since it requires us to initiate the algorithm at a vertex of the simplex and to discard the information we have already obtained as to the approximate location of the equilibrium price vector. Alternatively there are numerical

methods--essentially adaptations of Newton's method to systems of nonlinear inequalities--which can be used to improve the current approximation. But neither of these approaches are remotely as good as the algorithms of Merrill and Eaves which are described in the next section.

IX. THE ALGORITHMS OF MERRILL AND EAVES

The computational methods we have discussed in the previous sections have two major drawbacks. First of all they require the algorithm to be initiated at a vertex of the unit simplex--or as in Kuhn's version of integer labeling at a boundary point of the simplex. If an answer is obtained with a fixed grid, whose accuracy is inadequate for the problem at hand, the algorithm must be restarted with a finer grid and the results of the previous calculations discarded completely. The algorithms introduced by Merrill [43] and Eaves [11] permit the computation to be initiated at an arbitrary point on the simplex and allow a continual refinement of the grid. They yield a vast improvement in computational speed over the earlier algorithms which require a fixed simplicial decomposition, and are used in virtually all practical applications of fixed point methods.

Merrill's method, which is essentially identical to the "sandwich" method subsequently introduced by Kuhn [36], is particularly simple to describe. In order to introduce the basic idea of Merrill's method, let us consider the case in which the unit simplex is 1-dimensional and where the basic problem is to be solved by integer labeling techniques. The interval $\{(x_1, x_2) | x_i \geq 0,\ x_1 + x_2 = 1\}$ is subdivided with some preassigned grid size D. The vertices in the subdivision will be of the form $(k_1/D, k_2/D)$ with k_1 and k_2 non-negative integers summing to D. The numerators of the vertices of a typical simplex in the subdivision will be given by

$$\begin{bmatrix} k_1 & k_1 + 1 \\ k_2 & k_2 - 1 \end{bmatrix}.$$

The vertex defined by $(0,D)$ will be given the label 1 and $(D,0)$ the label 2 . The customary integer labeling methods starts at one of these extreme vertices and moves into the simplex until we first encounter a vertex with the other label.

Let us imagine that we have an initial guess as to where the true answer lies, given by the vector $(k_1^*/D,\ k_2^*/D)$, with k_1^* , k_2^* non-negative integers adding to D . Merrill's algorithm incorporates this additional information by drawing the simplex $k_1 + k_2 = D-1$, and subdividing the resulting two dimensional figure.

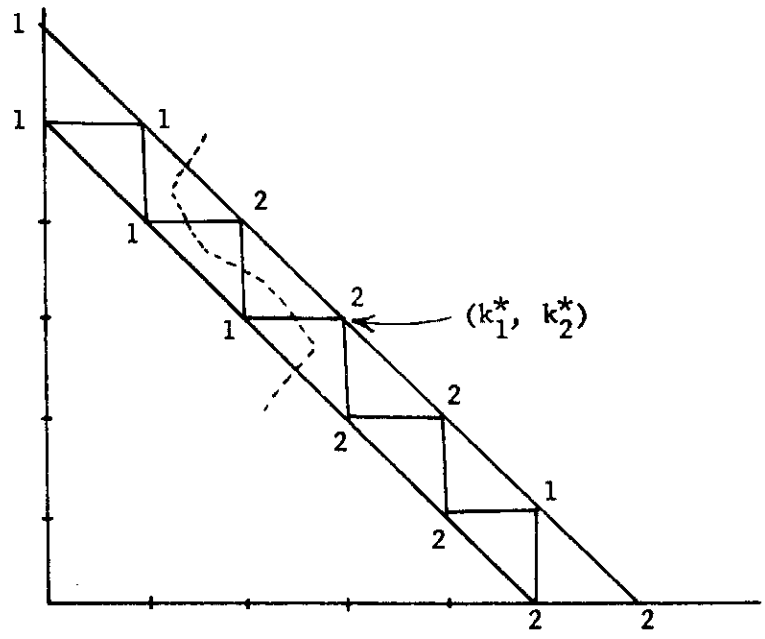

FIGURE 23

In Figure 23, $D = 6$, and the integer labels on the upper simplex are assumed to be derived from the problem at hand. The integer labels on the lower simplex will be based on the initial guess $(k_1^*,\ k_2^*)$ which in this particular example we assume to be $(3,3)$. On the lower level the pair $(k_1,\ k_2)$ will be given the label i if i is the first coordinate for which $k_i < k_i^*$.

Merrill's algorithm moves through a sequence of two-dimensional simplices, each defined by three vertices. We begin at the simplex

$$\begin{bmatrix} k_1^* & k_1^* - 1 & k_1^* \\ k_2^* & k_2^* & k_2^* - 1 \end{bmatrix}$$

two of whose vertices are on the artificial level and one vertex on the original simplex. The initial simplex has been constructed in such a way that the two vertices on the artificial level bear the labels 1 and 2, one of which is shared with the vector (k_1^*, k_2^*). We remove that vertex on the artificial level with the doubled label and continue until we are forced to exit through the original face of the "sandwich." At that point we have found a completely labeled simplex whose labels derive from the original problem. If the accuracy which has been achieved is not adequate we repeat the calculation, with say a doubled grid size, and with the new guess given by a choice of one of the vertices in the completely labeled simplex. Since the grid size grows exponentially we can expect a high degree of precision in a relatively small number of iterations.

The layer which is subdivided is part of a two dimensional simplex, obtained by introducing an additional coordinate $k_0 = D - k_1 - k_2$. With this notation the initial simplex may be represented by

$$\begin{bmatrix} 0 & 1 & 1 \\ k_1^* & k_1^* - 1 & k_1^* \\ k_2^* & k_2^* & k_2^* - 1 \end{bmatrix}.$$

This permits us to make use of the regular simplicial subdivision based

on the vertices (k_0, k_1, k_2) with $\sum_0^2 k_i = D$, and in particular to expliot the simplicity of the replacement step. Of course, vectors on the artificial level will have $k_0 = 1$, and those on the original level $k_0 = 0$.

Let us generalize these observations, by considering a problem of size n, and by making use of vector labeling. We are concerned with the set of non-negative integers $(k_1, \ldots, k_n)$ summing to D. Each such vector has associated with it a vector label $\ell(k)$ in R^n, derived from the problem at hand, and satisfying the condition that $\ell(k)$ is the i^{th} unit vector in R^n if k_i is the first coordinate equal to zero. We wish to determine a simplex in the subdivision, such that a non-negative linear combination of the associated vector labels equals a preassigned positive vector b. In addition, we are given an initial guess $(k_1^*, \ldots, k_n^*)$ as to the solution.

The set of vertices is enlarged by considering an additional coordinate k_0 and the regular simplicial decomposition based on the set of vertices $(k_0, k_1, \ldots, k_n)$ with $k_i \geq 0$, $\sum_0^n k_i = D$. Only those vertices with $k_0 = 0, 1$ are relevant for the computation and these are associated with vector labels in R^n (not R^{n+1}) according to the following rule.

9.1. [labeling rule for Merrill's algorithm] A vector $k = (k_0, k_1, \ldots, k_n)$ will, if $k_0 = 0$, receive the label $\ell(k_1, \ldots, k_n)$. If $k_0 = 1$ the vector label is the i^{th} unit vector in R^n if i is the first coordinate for which $k_i < k_i^*$.

The initial simplex is given by the columns of the following $(n+1) \times (n+1)$ matrix

$$\begin{bmatrix} 0 & 1 & \cdots & 1 \\ k_1^* & k_1^* - 1 & & k_1^* \\ \vdots & \vdots & & \vdots \\ k_n^* & k_n^* & & k_n^* - 1 \end{bmatrix}.$$

If the rows and columns of this matrix are indexed from $0, \ldots, n$, we see that columns $1, \ldots, n$ have as vector labels the n unit vectors in R^n. These vector labels form a feasible basis for the system $Ly = b$. The vector label associated with the zeroth column derives from the original problem.

The first step of the algorithm is to introduce this latter vector into the feasible basis, by a pivot step. One of the other vectors is eliminated and its replacement found by determining an adjacent simplex. The vector label associated with the new vertex is brought into the feasible basis and we continue.

The algorithm terminates when the simplex contains n vectors whose coordinate is 0 (i.e. vectors whose labels come from the original problem) and a single vector whose zeroth coordinate is 1 and whose associated column has just been removed from the feasible basis by a pivot step. The labels associated with these n vectors therefore form a feasible basis for $Ly = b$, and we have obtained a solution to the original problem. If the accuracy is not adequate the algorithm is repeated with a higher value of D.

The argument for the second algorithm of Section VI can be used to show that we never depart from the "sandwich" other than through the top face. Demonstrating that the algorithm cannot cycle is also a simple application of the types of analysis used in previous sections.

Merrill's method may be viewed in terms of homotopy theory. We take the product of the unit simplex and the closed unit interval $[0,1]$ and subject it to a simplicial subdivision all of whose vertices lie on one of the two bounding planes. We construct a piecewise linear mapping of this product into the unit simplex which approximates the true mapping on the face $x_0 = 0$. On the face $x_0 = 1$ the mapping is chosen so as to have a unique fixed point. Merrill's method essentially traces the fixed point from one face to another.

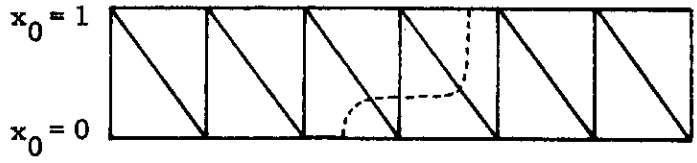

FIGURE 24

Eaves' method, on the other hand, involves a simplicial decomposition of the product of the unit simplex and the half line $[0,\infty)$ with the grid becoming finer as the coordinate x_0 increases. Eave's method has

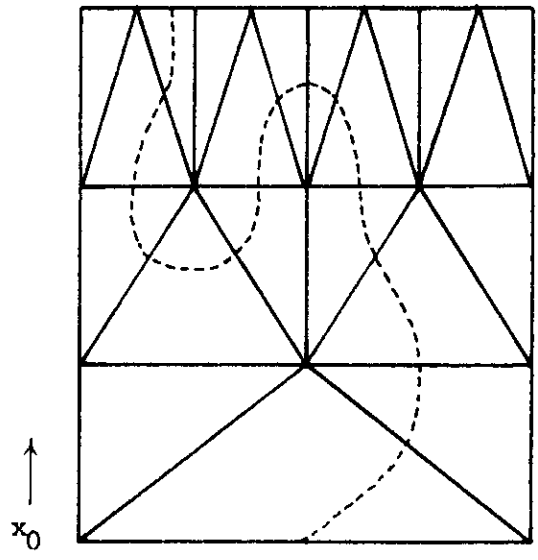

FIGURE 25

the distinct advantage that the vector labels obtained at the previous level are retained. This means that the Jacobian of the mapping is utilized rather than a single estimate of the fixed point. Eaves' method does, however, require the construction of a simplicial decomposition of the product space with exponentially decreasing grid size and this is rather complicated to work with.

In applying Merrill's algorithm to the general equilibrium model of the last section we began with a small value of D and an initial estimate of $k^* = (9, 9, 8, 8, 8)$. The grid size was then tripled a total of nine successive times. In roughly 2000 iterations--essentially the same as that required by a fixed grid size of 200--the following simplex was obtained.

$$\begin{bmatrix} 216827 & 216827 & 216827 & 216827 & 216827 & 216828 \\ 247077 & 247077 & 247077 & 247078 & 247078 & 247077 \\ 158471 & 158472 & 158472 & 158471 & 158471 & 158471 \\ 54073 & 54072 & 54072 & 54072 & 54073 & 54073 \\ 104398 & 104398 & 104399 & 104399 & 104398 & 104398 \\ 203304 & 203304 & 203303 & 203303 & 203303 & 203303 \end{bmatrix}$$

Any of these columns can be taken as an estimate of the equilibrium price vector, after division by the sum of the entries in the column

Equilibrium Prices

.220319	.251057	.161024	.054943	.106080	.206578

These prices may then be used to calculate the profits in the following table in which the activity levels are derived from the weights in the final feasible basis.

Activity	Level	Profit
7	0.463533	-0.000003
8	0.0	-0.141668
9	3.939607	-0.000001
10	0.006050	-0.000002
11	0.0	-0.007615
12	0.0	-0.052769
13	0.438389	-0.000006
14	0.0	-0.254323

The final table illustrates the relationship between supply and demand based on these prices and activity levels. As can be seen the degree of approximation has been substantially improved--with no increase in computational cost--over the answer based on a fixed grid size.

Commodity	Demand	Supply
1	11.234238	11.235904
2	0.0	-0.001595
3	1.155703	1.154355
4	2.974338	2.973208
5	23.682987	23.686038
6	9.354235	9.354134

BIBLIOGRAPHY

[1] ALLGOWER, E. L., Application of a Fixed Point Search Algorithms to Nonlinear Problems Having Several Solutions, Proceedings of Conference on Computing Fixed Points with Applications (1974).

[2] _____ and M. M. JEPPSON, The Approximation of Solutions of Nonlinear Elliptic Boundary Value Problems with Several Solutions, Springer Lecture Notes 333 (1973), 1-20.

[3] COHEN, D. I. A., On the Sperner Lemma, J. Comb. Theory, 2 (1967), 585-587.

[4] COTTLE, R. W. and G. B. DANTZIG, A Generalization of the Linear Complementarity Problem, J. Comb. Theory, 8, 1 (1970), 79-90.

[5] DANTZIG, G. B. and A. S. MANNE, A Complementarity Algorithm for an Optimal Capital Path with Invariant Proportions, International Institute for Applied Systems Analysis, Laxenburg, Austria, Revised (1974).

[6] DEBREU, G., Excess Demand Functions, Journal of Mathematical Economics, 1 (1974), 15-21.

[7] _____, Four Aspects of the Mathematical Theory of Economic Equilibrium, Proceedings of Int. Congress of Mathematicians, Vancouver (1974), 65-77.

[8] DOHMEN, J. and J. SCHOEBER, Approximated Fixed Points, EIT 55, Department of Econometrics, Tilburg University, Holland (1975).

[9] EAVES, B. C., On the Basic Theory of Complementarity, Mathematical Programming, 1, 1 (1971), 68-75.

[10] _____, Computing Kakutani Fixed Points, SIAM J. of Appl. Math., 21, 2 (1971), 236-244.

[11] _____, Homotopies for Computation of Fixed Points, Mathematical Programming, 3, 1 (1972), 1-22.

[12] _____, An Odd Theorem, Proc. AMS, 26, 3 (1970), 509-513.

[13] _____ and R. SAIGAL, Homotopies for Computation of Fixed Points on Unbounded Regions, Mathematical Programming, 3, 2 (1972), 225-237.

[14] EAVES, B. C. and H. SCARF, The Solution of Systems Piecewise Linear Equations, Math. of O.R., 1, 1 (February 1976), 1-27.

[15] FISHER, M. L. and F. J. GOULD, A Simplicial Algorithm for the Nonlinear Complementarity Problem, Mathematical Programming, 6, 3 (1974), 281-300.

[16] FISHER, M. L., F. J. GOULD and J. W. TOLLE, A New Simplicial Approximation Algorithm with Restarts: Relations between Convergence and Labelling, in Proc. Conf. on Computing Fixed Points with Applications, Department of Mathematical Sciences, Clemson University, Clemson, S.C. (1974).

[17] FREIDENFELDS, J., A Set Intersection Theorem and Applications, Mathematical Programming, 7, 2 (1974), 199-211.

[18] GARCIA, C. B., Continuation Methods for Simplicial Mappings, University of Chicago (1975).

[19] _____, A Hybrid Algorithm for the Computation of Fixed Points, Management Science, 22, 5 (January 1976), 606-613.

[20] _____ and W. I. ZANGWILL, Determining All Solutions to Certain Systems of Nonlinear Equations, Grad. School of Business, University of Chicago, December 1976, pp. 31.

[21] GINSBURGH, V. and J. WAELBROECK, A General Equilibrium Model of World Trade, Part I, Full Format Computation of Economic Equilibria, Cowles Discussion Paper No. 412, November 18, 1975, Yale University, pp. 37.

[22] _____, Computational Experience with a Large General Equilibrium Model, CORE Discussion Paper No. 7420, Heverlee, Belgium (1974), 21.pp.

[23] HANSEN, T., On the Approximation of a Competitive Equilibrium, Ph.D. Thesis, Yale University (1968).

[24] _____, On the Approximation of Nash Equilibrium Points in an N-Person Noncooperative Game, SIAM J. Appl. Math., 26, 3 (1974), 622-637.

[25] _____, and T. C. KOOPMANS, On the Definition and Computation of a Capital Stock Invariant under Optimization, J. Economic Theory, 5, 3 (1972), 487 523.

[26] HANSEN, T. and H. SCARF, On the Applications of a Recent Combinatorial Algorithm, Cowles Foundation Discussion Paper No. 272, Yale University (1969).

[27] HIRSCH, M. W., A Proof of the Nonretractibility of a Cell onto Its Boundary, Proc. of AMS, 14 (1963), 364-365.

[28] JEPPSON, M. M., A Search for the Fixed Points of a Continuous Mapping, Mathematical Topics in Economic Theory and Computation, ed. R. H. Day and S. M. Robinson (1972), 122-129.

[29] KANNAI, Y., On Closed Coverings of Simplexes, SIAM J. Appl. Math., 19, 2 (1970), 459-461.

[30] KARAMARDIAN, S. (Editor), Fixed Points: Algorithms and Applications, Proc. Conf. on Computing Fixed Points with Applications, Clemson University, Clemson, S.C. (1975).

[31] _____, The Nonlinear Complementarity Problem with Applications, Parts I and II, J. Optimization Theory and Applications, 4, 2 and 3 (1969), 87-98, 167-181.

[32] KELLOGG, R. B., T. Y. LI and J. YORKE, A Method of Continuation for Calculating a Brouwer Fixed Point, in S. Karamadian, ed., Computing Fixed Points with Applications, New York, Academic Press, 1977.

[33] KOJIMA, A., A Unification of the Existence Theorems of the Nonlinear Complementarity Problem, Math. Programming, 9, 3 (1975), 257-277.

[34] KUHN, H. W., Some Combinatorial Lemmas in Topology, IBM J. Research and Develop., 4, 5 (1960), 518-524.

[35] _____, Simplicial Approximation of Fixed Points, Proc. Nat. Acad. Sci., U.S.A., 61 (1968), 1238-1242.

[36] _____ and J. G. MACKINNON, The Sandwich Method for Finding Fixed Points, Department of Economics and Mathematics, Princeton University Technical Report (1975).

[37] LEMKE, C. E., Bimatrix Equilibrium Points and Mathematical Programming, Management Sciences, 11, 7 (1965), 681-689.

[38] _____ and J. T. HOWSON., JR., Equilibrium Points of Bimatrix Games, SIAM J. on Appl. Math., 12, 2 (1964), 413-423.

[39] MCFADDEN, D., A. MAS-COLELL, R. MANTEL, and M. K. RICHTER, A Characterization of Community Excess Demand Functions, Journal of Economic Theory, 9 (1975), 361-374.

[40] MACKINNON, J. G., Solving Economic General Equilibrium Models by the Sandwich Method, Proc. Conf. Computing Fixed Points with Applications, Department of Mathematical Sciences, Clemson University, Clemson, S.C. (1975).

[41] MANTEL, R. R., Toward a Constructive Proof of the Existence of Equilibrium in a Competitive Economy, Yale Economic Essays, 8 (1968), 155-200.

[42] _____, On the Characterization of Aggregate Excess Demand, Journal of Economic Theory, 9 (1974), 348-353.

[43] MERRILL, O. H., Applications and Extensions of an Algorithm that Computes Fixed Points of Certain Upper Semi-Continuous Point to Set Mappings, Ph.D. Dissertation, Department of Industrial Engineering, University of Michigan (1972), 228 pp.

[44] SAIGAL, R., On Paths Generated by Fixed Point Algorithms, Mathematics of Operations Research, 1, 4 (1976), 359-380.

[45] _____ and C. B. SIMON, Generic Properties of the Complementarity Problem, Mathematical Programming, 4, 3 (1973), 324-335.

[46] SCARF, H., The Approximation of Fixed Points of a Continuous Mapping, SIAM J. Appl. Math., 15, 5 (1967), 1328-1343.

[47] _____, On the Computation of Equilibrium Prices, in Ten Economic Studies in the Tradition of Irving Fisher, John Wiley, New York (1967).

[48] _____, The Core of an N Person Game, Econometrica, 35, 1 (1967), 50-69.

[49] _____ (with the collaboration of T. HANSEN), Computation of Economic Equilibria, Yale University Press, New Haven (1973), 249 pp.

[50] SHAPLEY, L. S., A Note on the Lemke-Howson Algorithm, Mathematical Programming Study, 1 (1974), 175-189.

[51] SHOVEN, J. B. and J. WHALLEY, A General Equilibrium Calculation of the Effects of Differential Taxation of Income from Capital in the U.S., J. Public Economics, 1 (1972), 281-321.

[52] SMALE, S., A Convergent Process of Price Adjustment and Global Newton Methods, Journal of Mathematical Economics, 3 (1976), 107-120.

[53] SONNENSCHEIN, H., Do Walras' Identity and Continuity Characterize the Class of Community Excess Demand Funttions?, Journal of Economic Theory, 6 (1973), 345-354.

[54] TODD, M. J., The Computation of Fixed Points and Applications, Lecture Notes in Economics and Mathematical Systems No. 124 (Mathematical Economics), Springer-Verlag, New York, 1976, pp. 129.

[55] _____, Orientation in Complementary Pivot Algorithms, Mathematics of Operations Research, 1, 1 (February 1976), 54-66.

[56] _____, On Triangulations for Computing Fixed Points, Mathematical Programming, 10 (1976), 322-346.

[57] WILMUTH, R. J., The Computations of Fixed Points, Department of Operations Research, Stanford University, Ph.D. Thesis (1973).

[58] WILSON, R., Computing Equilibria of N-Person Games, SIAM J. Appl. Math., 21, 1 (1971), 80-87.