SOME TOPICS IN TWO-PERSON GAMES

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INTRODUCTION

This note reports on half-a-dozen loosely related excursions into the theory of finite, two-person games, both zero-sum and non-zero-sum. The connecting thread is a general predilection for results that do not depend upon the full linear structure of the real numbers. Thus, most of our theorems and examples are invariant under order-preserving transformations applied to the payoff spaces, while a few (in § 1) are invariant under the group of transformations that commute with multiplication by -1.

We should make it clear that we are not interested in "ordinal" utility, as such, but rather the ordinal properties of "cardinal" utility. The former would require a conceptual reorientation which we do not wish to undertake here. Nevertheless, the ordinalist may find useful ideas in this paper.

Rather than summarize the whole paper here, we shall merely take a sample; for a more synoptic view the reader is invited to scan the section headings. Consider first the matrix game shown in the margin. The solution is easily found as soon as we recognize that the game is symmetric in the players. Indeed, if we map each player's i^{th} strategy into the other's $i+1^{st}$ (mod 4), we merely reverse the signs of the payoffs. It follows that the value is 0 and that there is a solution of the form (a,b,a,b), (b,a,b,a). (See § 1.)

0	1	2	−1
−1	0	1	−2
2	−1	0	1
1	−2	−1	0

Next, consider the class of matrix games in which the payoffs are ordered like those in the matrix at left (next page). In all these games, player I's third strategy is never playable, although it is not dominated in the usual sense. To verify this, observe that if the value of the 2-by-2

10	0	8
1	9	7
2	3	6
3	4	5

subgame in the upper left corner is greater than "3" the third row is dominated by a linear combination of the first two rows, while if it is less than "7" the third column can be dropped, and then the third row. (See § 3.)

Finally, consider the non-zero-sum game with outcome matrix as shown at right. Player I rates the outcomes A > B > C; player II rates them B > A > C. If we apply the algorithm of "fictitious play" to this game, a strange thing happens. Rather than converging to the unique equilibrium point (at which all probabilities are equal), the sequence of mixed-strategy pairs generated by the algorithm oscillates around it, keeping a finite distance away. (See § 5.)

A	C	B
B	A	C
C	B	A

The five main sections of this paper are essentially independent, both logically and topically. Our reason for combining them into a single paper is the hope that they will appeal to a single audience. Much of this work has already appeared in short RAND Memoranda [15], and some of it has been cited in the published literature [5], [11]. In reworking this material, however, we have added many new results.

§ 1. SYMMETRIC GAMES

1.1. Discussion

It is easy to see that a two-person zero-sum game can be symmetric in the players without having a skew-symmetric payoff matrix. "Matching Pennies" is a simple example; another is shown at the right; and another is given in the Introduction. The point is, of course, that an automorphism of the game that permutes the players can simultaneously shuffle the labels of the pure strategies. It would be interesting to know something about the abstract structure of such automorphisms.*

1	1	−1
−1	0	−1
−1	1	1

As a first step, we shall show that the matrix of a symmetric game can be decomposed into an array of square blocks in such a way that (a) each block has constant diagonals (in one direction), (b) the array as a whole is skew-symmetric in a certain sense, and (c) the size of each block is a power

* The narrow "skew-symmetric" definition is most often seen in the literature (e.g., [19], [10], [7]). But Nash uses the more general form in [13]; see also [19], p. 166.

	1	3	2
1	1	−1	1
3	−1	1	1
2	−1	−1	0

of 2. This is illustrated at left for the 3-by-3 example given above.

The "power of 2" property, (c), is quite interesting. It tells us, for example, that the 6-by-6 game illustrated below, which is obviously symmetric and has constant diagonals, can nevertheless be decomposed into smaller blocks. The 4-by-4 and 8-by-8 analogues of this matrix, on the other hand, do not decompose.

	1	2	3	4	5	6
1	1	2	3	−3	−2	−1
2	−1	1	2	3	−3	−2
3	−2	−1	1	2	3	−3
4	−3	−2	−1	1	2	3
5	3	−3	−2	−1	1	2
6	2	3	−3	−2	−1	1

$\longrightarrow$

	2	5	3	6	4	1
1	2	−2	3	−1	−3	1
4	−2	2	−1	3	1	−3
2	1	−3	2	−2	3	−1
5	−3	1	−2	2	−1	3
3	−1	3	1	−3	2	−2
6	3	−1	−3	1	−2	2

1.2. The Main Theorem

Let $A, B, \ldots$ denote n-by-n game matrices (n fixed throughout), and let $P, Q, R, \ldots$ denote permutation matrices of the same size. Primes will denote transposition, which is equivalent to inversion for permutation matrices. We define the following matrix properties:

equivalence: $A \equiv B \longleftrightarrow A = PBQ'$ for some P, Q.

symmetry: $A \in \Sigma \longleftrightarrow A \equiv -A'$.

conjugates: $P \approx Q \longleftrightarrow P = RQR'$ for some R.

Certain subclasses of Σ will also be of interest:

$$A \in \Sigma(P,Q) \longleftrightarrow B = -PB'Q' \quad \text{for some} \quad B \equiv A.$$

These subclasses exhaust Σ, but are not disjoint. Indeed, we have

LEMMA 1. If $PQ \approx RS$ then $\Sigma(P,Q) = \Sigma(R,S)$.

PROOF. Given $PQ = TRST'$ and $A = -PA'Q'$, we require $B \equiv A$ such that $B = -RB'S'$. As it happens, $B = T'AP'TR$ serves the purpose. Indeed, since $P' = QTS'R'T'$, we have

$$\begin{aligned} B &= T'(-PA'Q')(QTS'R'T')TR \\ &= -T'PA'TS' \\ &= -(T'AP'T)'S' \\ &= -RB'S'. \end{aligned}$$

Note that $\Sigma(P,Q)$ depends only on PQ. If we define $\Sigma(P) = \Sigma(P,I)$, where I is the identity, then Lemma 1 may be restated:

LEMMA 1'. If $P \approx Q$ then $\Sigma(P) = \Sigma(Q)$.

It can be shown that the converse is valid. Since two permutations are conjugate if and only if their cyclic factors have matching periods, there is thus a one-to-one correspondence between the classes $\Sigma(P)$ and the partitions of n. This stronger result is not used in what follows.

LEMMA 2. If Q is an odd power of P then $\Sigma(Q) \supseteq \Sigma(P)$.

PROOF. Suppose $A = -PA'$. We must show that $A \in \Sigma(P^{2k+1})$ for all k. But $A = -P(-PA')' = PAP'$; hence $A = P^kAP'^k = -P^{k+1}A'P'^k$. Therefore $A \in \Sigma(P^{k+1}, P^k) = \Sigma(P^{2k+1})$.

THEOREM 1.1. Every symmetric game $A \in \Sigma$ is equivalent to a game B satisfying $B = -RB'$ for some permutation R, the order of which is a power of 2.

PROOF. Let $A \in \Sigma(P)$. The order of P can be represented in the form $c2^k$, with c odd. Then the order of P^c will be 2^k. By Lemma 2, $A \in \Sigma(P^c)$. Thus P^c will serve as the R of the theorem. Q.E.D.

1.3. Block Decomposition

The decomposition into blocks can now be described. By proper choice of B, we can give R the form:

$$R = (1\ 2\ 3\ \ldots\ \lambda_1)(\lambda_1+1\ \ldots\ \lambda_1+\lambda_2)\ \ldots\ (\ldots\ n),$$

where the periods λ_μ are powers of 2. B now breaks up into a square array of λ_μ-by-λ_ν blocks $B_{\mu\nu}$, and the equation $B = -RB'$ implies (and is implied by) the structure set forth in the following theorem. The proof is straightforward.

THEOREM 1.2. Let $\lambda = \min(\lambda_\mu, \lambda_\nu)$. The block $B_{\mu\nu}$ has constant diagonals, in the sense that numbers $\beta_1, \ldots, \beta_\lambda$ exist such that for $\overline{ij}$ in that block,*

$$b_{ij} = \beta_h \quad \text{if} \quad i - j \equiv h \pmod{\lambda}.$$

(If $\lambda_\mu \neq \lambda_\nu$, then $B_{\mu\nu}$ breaks up into identical square sub-blocks of size λ.) In the symmetrically located block $B_{\nu\mu}$ the same numbers appear; we have

$$b_{ij} = -\beta_{\lambda-h+1} \quad \text{if} \quad i - j \equiv h \pmod{\lambda}.$$

In particular, along the main diagonal of the array ($\mu = \nu$), we have $\beta_h = -\beta_{\lambda-h+1}$ for $h = 1, \ldots, \lambda$.

COROLLARY 1. Indecomposable symmetric n-by-n games exist only for n a power of 2.

COROLLARY 2. Every symmetric game of odd size has a zero in its payoff matrix.

1.4. Solutions

To solve a symmetric game we may (a) replace the $B_{\mu\nu}$ by their

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* We write $\overline{ij}$ for the ordered pair (i, j).

average values, (b) solve the resulting skew-symmetric matrix,* and (c) distribute the mixed-strategy probabilities for each block equally among its constituent pure strategies.** Not every solution of the original can be obtained in this way, however. In the game at the right, for example, (0, 2/3, 1/3, 0) is a basic (extreme) optimal strategy of each player.

1	−1	2	−2
−1	1	−2	2
2	−2	4	−4
−2	2	−4	4

1.5. Symmetric Nonzero-sum Games

There is a direct extension to nonzero-sum games. Let us call the matrix pairs (A_1, A_2) and (B_1, B_2) equivalent if, for some P, Q, both $A_1 = PB_1Q'$ and $A_2 = PB_2Q'$. Let us call (A_1, A_2) symmetric if (A_1, A_2) and (A_2', A_1') are equivalent. Then the following counterpart to Theorem 1.1 can be established by essentially the same proof:

THEOREM 1.3. Every symmetric nonzero-sum games (A_1, A_2) is equivalent to a game of the form $(B_1, B_2) = (RB_2', RB_1')$ for some permutation R of order a power of 2.

The description of the block decomposition remains much as before, though the "main diagonal" loses some of its special significance. Corollary 1 remains valid, but not Corollary 2.

§ 2. SOME THEOREMS ABOUT SADDLEPOINTS

2.1. A Condition for the Existence of a Saddlepoint

THEOREM 2.1. If A is the matrix of a zero-sum two-person game, and if every 2-by-2 submatrix of A has a saddlepoint, then A has a saddlepoint.

PROOF. Let val[A] = v. Let j be the index of a column having the minimum number, π, of entries greater than v. Suppose $\pi > 0$; then, for

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* See Kaplansky [10] and Gale, Kuhn, and Tucker [7].

** See for example, Gale, Kuhn, and Tucker [8], application (e).

some i we have $a_{ij} > v$. Since the value of the game is only v, we must have $a_{ij'} \leq v$ for some j'. But the column indexed by j' has at least π entries greater than v, too many to be paired off against the $\pi-1$ remaining entries $> v$ of the other column. Thus, for some i' we have $a_{i'j'} > v \geq a_{i'j}$. Since the 2-by-2 submatrix:

	j	j'
i	$> v$	$\leq v$
i'	$\leq v$	$> v$

has no saddlepoint, the assumption $\pi > 0$ was incorrect. Hence there is a column with no entries greater than v. Similarly there is a row with no entries less than v. Q.E.D.

2.2. Detached Rows and Columns

The hypothesis of Theorem 2.1 actually imposes a very special structure on the matrix A. Let us say that the p^{th} row of A is detached if

$$\max_j a_{pj} \leq \max_{i \neq p} \min_j a_{ij}.$$

Similarly, the q^{th} column is detached if

$$\min_i a_{iq} \geq \min_{j \neq q} \max_i a_{ij}.$$

Detachment obviously implies domination. For 2-by-2 matrices, the existence of a detached row or column is equivalent to the existence of a saddlepoint.

THEOREM 2.2. If every 2-by-2 submatrix of A has a saddlepoint, then A has a detached row or column.

PROOF. By Theorem 2.1, both A and A' (its transpose) have saddlepoints. Hence there is a column of A with no entries greater than val[A], as well as a column of A (row of A') with no entries less than val[A']. If val[A] < val[A'], then these columns are distinct, and the latter is a detached column. Similarly, if val[A] > val[A'], there is a detached row. If val[A] = val[A'] = v, then there is either a detached row or column, or a saddlepoint $\overline{pq}$ common to both A and A'. In the latter case, $a_{pj} \equiv a_{iq} \equiv v$, all i, j, and we can use the fact that the submatrix obtained by deleting row p and column q has a saddlepoint to show that either p or q is detached, depending on whether the value of the submatrix is $\geq v$ or $\leq v$. Q.E.D.

2.3. A Generalization

Theorem 2.1 can be generalized, after a fashion. Let us say that a matrix is "in general position" if no two collinear entries are equal.

THEOREM 2.3. Let A be an m-by-n matrix in general position, and let $2 \leq r \leq m$, $2 \leq s \leq n$. If every r-by-s submatrix of A has a saddlepoint, then A has a saddlepoint.

PROOF. It suffices to prove the case where $r = m \geq 2$ and $s = n-1 \geq 2$; the rest will follow by induction and symmetry. Let A^q denote A with the q^{th} column deleted. Let $\overline{i_q j_q}$ be the location of the saddlepoint of A^q (which is unique, since A is in general position). If all of the j_q's are distinct, for $q = 1, \ldots, n$, then every column of A will contain one of the points $\overline{i_q j_q}$. Since each $a_{i_q j_q}$ is a column maximum, one of them will be the maximum of the whole matrix. On the other hand, it must also be the (strict) minimum of its row in A^q, which contains at least two entries. This impossibility implies that the j_q's are not all distinct. Let $j_q = j_t$, $q \neq t$. Then it is apparent that the point $\overline{i_q j_q} = \overline{i_t j_t}$ is a saddlepoint of A. Q.E.D.

At right we illustrate what can happen if two collinear entries are equal. Every 3-by-2 submatrix has a saddlepoint, but not the full matrix.

1	0	−1
−2	0	2
2	−1	3

§ 3. ORDER MATRICES

3.1. Definitions. The Saddle

By a line of a matrix we shall mean either a row or a column. Two numerical matrices of the same size will be called order equivalent if the elements of corresponding lines are ordered alike. An order matrix, $\mathfrak{A}$, is an equivalence class of order-equivalent numerical matrices. Abstractly, it may be regarded as a partial ordering $\underset{\mathfrak{A}}{<}$ on the set $I(\mathfrak{A})$ of all index pairs $\overline{ij}$, with the property that collinear points are always comparable, while noncollinear points are never comparable, except as a result of transitivity.

If $K \subseteq I(\mathfrak{A})$ is a set of index pairs, we shall write K_1 for the set of first members, K_2 for the set of second members, and $\overline{K}$ for $K_1 \times K_2$. In other words, K_1 is the smallest set of rows, K_2 the smallest set of columns, and $\overline{K}$ the smallest submatrix, that covers K. If $K = \overline{K}$ then K is rectangular.

A generalized saddle point (GSP) of an order matrix $\mathfrak{A}$ is a rectangular set $K \subseteq I(\mathfrak{A})$ such that (1) for each $i \notin K_1$ there is a $p \in K_1$ with $\overline{pj} \underset{\mathfrak{A}}{>} \overline{ij}$ for all $j \in K_2$, and (2) for each $j \notin K_2$ there is a $q \in K_2$ with $\overline{iq} \underset{\mathfrak{A}}{<} \overline{ij}$ for all $i \in K_1$.* (Note the strict inequalities.) A GSP that contains no other GSP is called a saddle.

THEOREM 3.1. Every order matrix has a unique saddle.

The proof consists in showing that the intersection of two GSP's is a GSP. Let K and L be two GSPs of $\mathfrak{A}$. Then certainly $K \cap L \neq \phi$, since K and L must both carry the optimal mixed strategies of any numerical matrix belonging to $\mathfrak{A}$.** It will suffice to show that for each $i \notin K_1$ there is a p in $K_1 \cap L_1$ with $\overline{pj} > \overline{ij}$ for all $j \in K_2 \cap L_2$. But we can find a $p' \in K_1$ with that property, since K is a GSP. If $p' \notin L_1$, then we can find a $p'' \in L_1$ with $\overline{p''j} \underset{\mathfrak{A}}{>} \overline{p'j}$ for all $j \in K_2 \cap L_2$. If $p'' \notin K_1$, then $p''' \in K_1$ can be found bearing the same relation to p'', etc. The strict inequalities ensure that the sequence terminates. That is, there is a row in $K_1 \cap L_1$ that majorizes row i on $K_2 \cap L_2$. Q.E.D.

The saddle of $\mathfrak{A}$ will be denoted by $S(\mathfrak{A})$. It contains everything

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* Compare [5], pp. 41-42.

** Bass [1] found a way to prove $K \cap L \neq \phi$ directly, without reference to optimal strategies.

relevant to the solutions of the numerical games A belonging to $\mathfrak{A}$. In particular, it contains Bohnenblust's "essential" submatrix* and the Shapley-Snow "kernels."**

We may define a weak GSP by using nonstrict inequalities $\underset{\mathfrak{A}}{\geq}$, $\underset{\mathfrak{A}}{\leq}$. The intersection of two weak GSP's need not be a weak GSP. However, as Bass has shown, the intersection of a weak GSP with the saddle is a weak GSP.

3.2. Residuals

A one-element saddle is an ordinary (strict) saddlepoint, and is easy to find, even in a very large matrix. Identification of the saddle in general may be more tedious. The next theorem, based on an ingenious idea of Harlan Mills, provides us a rapid method of generating points in the saddle, and thereby substantially reduces the search. (See also § 3.5.)

LEMMA 1. (Mills) Let $\mathfrak{A}$ be an order matrix with at least two columns, let $\overline{pq}$ be maximal with respect to $\underset{\mathfrak{A}}{<}$, and let $\mathfrak{A}^q$ be obtained from $\mathfrak{A}$ by deleting column q.*** Then $S(\mathfrak{A}^q) \subseteq S(\mathfrak{A})$.

PROOF. Write S for $S(\mathfrak{A})$ and S^q for $S_1 \times (S_2 - \{q\})$. If $q \notin S_2$, then S^q is at once seen to be a GSP of $\mathfrak{A}^q$. If $q \in S_2$, then $p \in S_1$ by maximality. To show that S^q is a GSP of $\mathfrak{A}^q$, note that the row condition is satisfied for S^q, just as for S, while the column condition fails only if column q was used (in $\mathfrak{A}$) to minorize some column outside S_2. But this is impossible, by the maximality of $\overline{pq}$. Thus S^q is always a GSP of $\mathfrak{A}^q$. Hence $S(\mathfrak{A}^q) \subseteq S^q \subseteq S$.

A similar lemma holds concerning the deletion of rows containing minimal elements.

The following lemma is easily established; we omit the proof.

LEMMA 2. If $\mathfrak{B}$ is obtained from $\mathfrak{A}$ by deletion of a strictly majorized row or a strictly minorized column,

* See [2], p. 52; also [9], p. 44.

** See [17], p. 32.

*** I.e., delete column q from any $A \in \mathfrak{A}$, then use order equivalence.

then $S(\mathcal{B}) \subseteq S(\mathcal{A})$.

Now consider a sequence of nonvoid order matrices $\mathcal{A}$, $\mathcal{A}^{(1)}$, $\mathcal{A}^{(2)}$, ..., each obtained from the preceding by deleting a column containing a maximal point, <u>or</u> a row containing a minimal point, <u>or</u> a strictly majorized row, <u>or</u> a strictly minorized column. If $\mathcal{A}$ is an m-by-n matrix, then $\mathcal{A}^{(m+n-2)}$ is 1-by-1. The point so defined is called a <u>residual</u> of $\mathcal{A}$. Let $R(\mathcal{A})$ denote the set of all residuals of $\mathcal{A}$. Our lemmas, together with the fact that the saddle is rectangular, give us the following result:

THEOREM 3.2. $R(\mathcal{A}) \subseteq \overline{R(\mathcal{A})} \subseteq S(\mathcal{A})$.

3.3. Examples

Let us illustrate the residual concept.

(a) In the order matrix of "Matching Pennies" (at left), every entry is either maximal or minimal. By the maximality of $\overline{11}$ we can strike out the first column, then the first row goes by the minimality of $\overline{12}$, and $\overline{22}$ is a residual. The three other sequences give us $\overline{11}$, $\overline{12}$, and $\overline{21}$ as residuals. Thus $R(\mathcal{A}) = \overline{R(\mathcal{A})} = S(\mathcal{A}) = I(\mathcal{A})$.

1	−1
−1	1

(b) More generally, let $\mathcal{A}$ be a square matrix with a "detached" diagonal; for example let $\overline{ii}$ be maximal with respect to $\underset{\mathcal{A}}{<}$, for all i. Then for each i, we can delete all columns except i by Lemma 1, and then by row deletions obtain $\overline{ii}$ as a residual. Hence $\overline{R(\mathcal{A})} = S(\mathcal{A}) = I(\mathcal{A})$.

(c) We first conjectured [15b] that deletions of the first type (Lemma 1) would always suffice to determine the saddle. This was disproved by Bass [1] with the 4-by-4 example at right, which has no residuals "of the first type" in the top row. But if, for example, we strike out column 3, then row 3 goes by domination (Lemma 2), and the resulting 3-by-3 game has a detached diagonal, making $\overline{11}$ a residual by (b) above. (All residuals are circled.)

(1)	(11)	2	(13)
(9)	(6)	(3)	14
2	(5)	(15)	4
(8)	(12)	(10)	(7)

(d) Thus, a new counterexample is needed to show that $\overline{R(\mathcal{A})} \neq S(\mathcal{A})$ is possible. It is provided by the example described in the Introduction, which we reproduce here with residuals circled. As we shall see later, it is no coincidence that the residuals avoid the "unplayable" third row.

(10)	(0)	(8)
1	(9)	(7)
2	3	6
(3)	(4)	5

3.4. Minimax and Maximin Points

A minimax point of a numerical matrix is a column maximum of smallest value. A maximin point is a row minimum of largest value. Since noncollinear comparisons are usually involved, these terms are not well defined for order matrices. Nevertheless, we have the following:

THEOREM 3.3. The minimax and maximin points of any numerical matrix $A \in \mathfrak{A}$ are residuals of $\mathfrak{A}$.

PROOF. Let $\overline{pq}$ be a minimax point of some $A \in \mathfrak{A}$. If A has more than one column, then it will have a maximum in some column other than q, and the corresponding location in $\mathfrak{A}$ will be maximal. Delete this column and $\overline{pq}$ is still minimax. In this way we may strip the matrix down to just the single column q, and then by row deletions obtain $\overline{pq}$ as a residual. The proof for maximin points is similar.

Bass [1] has shown that every weak GSP contains a minimax point and a maximin point.

3.5. An Algorithm

The following method for determining the saddle was suggested by Mills (see [1], p. 3): Start with any submatrix K^0 known to be contained in the saddle (for example, a single residual). For $h = 0, 1, \ldots$, given $K^h \subseteq S(\mathfrak{A})$, construct K^{h+1} by adjoining either (a) a new row that is maximal (i.e., not strictly majorized) in $I_1(\mathfrak{A}) \times K_2^h$, or (b) a new column that is minimal in $K_1^h \times I_2(\mathfrak{A})$. Then $K^{h+1} \subseteq S(\mathfrak{A})$. If neither (a) nor (b) is possible, then $K^h = S(\mathfrak{A})$.

3.6. The Center

Let $C_1(A)$ denote the set of "active" strategies for the first player in A, i.e., those that appear with positive probability in some solution. Similarly, $C_2(A)$ for the second player. Let $C(A) = C_1(A) \times C_2(A)$. Then the set

$$C(\mathfrak{A}) = \bigcup_{A \in \mathfrak{A}} C(A)$$

will be called the <u>center</u> of $\mathfrak{A}$.* The center is contained in the saddle. It is not necessarily rectangular, as may be seen from the example at the right (center shaded).

0	4	2
4	0	3
1	2	5

Example (d) of § 3.3 suggests that a point which never occurs in optimal play cannot be a residual, i.e., that $R(\mathfrak{A}) \subseteq C(\mathfrak{A})$. This is not always true. For example, a matrix not "in general position" can cause trouble, as at left.

(0)	(1)	(1)
3	(0)	(2)

Another difficulty is more basic. In the matrix at right, columns 4 and 5 serve to establish that "10" < "13" and "15" < "18". These relations make row 1 unplayable, despite the presence of residuals.*

1	(10)	(4)	11	5
(15)	7	(6)	8	16
(2)	(18)	(3)	19	17
(13)	(9)	(14)	12	20

Accordingly, we modify the order matrix concept for the sequences $\{\mathfrak{A}^{(k)}\}$ that define $R(\mathfrak{A})$ (see § 3.2), retaining noncollinear comparisons from $\mathfrak{A}$ whenever applicable. The new relations can destroy maximal or minimal elements, but not create them; hence a <u>restricted residual set</u> $R^{\#}(\mathfrak{A}) \subseteq R(\mathfrak{A})$ results. Our aim is to prove that $R^{\#}(\mathfrak{A}) \subseteq C(\mathfrak{A})$, for $\mathfrak{A}$ in general position.

LEMMA 3. If $\overline{pq} \in C(A)$ then every neighborhood of A contains a B having a <u>unique</u> solution and such that $\overline{pq} \in C(B)$.

PROOF. If $\overline{pq} \in C(A)$ then a "basic" (extremal) solution (x^o, y^o) of A exists with $x^o_p > 0$, $y^o_q > 0$. We may assume that $\mathrm{val}[A] = v \neq 0$. By the Shapley-Snow theorem on basic solutions [17], a nonsingular submatrix $\dot{A}$ can be found for which $(\dot{x}^o, \dot{y}^o)$ is an "equalizer":

$$\dot{x}^o\dot{A} = \dot{y}^o\dot{A}' = v\dot{1}, \text{ whence } \dot{x}^o = v\dot{1}\dot{A}^{-1},\ \dot{y}^o = v\dot{1}\dot{A}'^{-1}.$$

(Dots indicate suppression of indices not appearing in $\dot{A}$; $\dot{1}$ is a vector of 1's.) Given $\epsilon > 0$, define $\dot{B} = \dot{A} - (\epsilon/v)\dot{A}^2$. If ϵ is sufficiently small, we can invert $\dot{B}$, as follows:

$$\dot{B}^{-1} = \dot{A}^{-1} + (\epsilon/v)\dot{I} + O(\epsilon^2).$$

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* Proof: If row 1 is active, then so is row 2, by an easy domination argument. Let $d = (7-15, 1-10, 10-7, 15-1)$. Then $\Sigma_i d_i = 0$, and we verify that $\Sigma_i a_{ij} d_i > 0$ for each j. Hence, any x with $x_1, x_2 > 0$ can be improved.

Then $\dot{B}$ has an equalizer solution proportional to $(\dot{x}^o, \dot{y}^o) + \epsilon(\dot{1}, \dot{1}) + O(\epsilon^2)$. This is all-positive for small ϵ, and hence unique.*

We now replace $\dot{A}$ by $\dot{B}$ in the original matrix. Since the value and solution of $\dot{B}$ differ only slightly from those of $\dot{A}$, it is easy to perturb the rows and columns outside the submatrix so as to make them irrelevant. The full matrix now has a unique solution, and p and q are still active. Q.E.D.

LEMMA 4. Let A have at least two columns, let A^q result from A by deleting column q, and let $A(\omega)$ result from A by adding ω to the entry at $\overline{pq}$. Then $\lim_{\omega\to\infty} \mathrm{val}[A(\omega)] = \mathrm{val}[A^q]$.

PROOF. Let x^o be optimal in A^q. Given $\epsilon > 0$, choose x such that $\|x - x^o\| \leq \epsilon$ and $x_p > 0$. Then

$$xA(\omega) \geq (v-\epsilon a,\ v-\epsilon a,\ \ldots,\ \underbrace{-a+\omega x_p}_{(q^{th}\text{ component})},\ \ldots,\ v-\epsilon a),$$

where $v = \mathrm{val}[A^q]$ and $a = \max_{ij}|a_{ij}|$. Thus, for ω large, x guarantees the first player at least $v - \epsilon a$ in the game $A(\omega)$. Hence we have

$$v - \epsilon a \leq \mathrm{val}[A(\omega)] \leq v. \qquad \text{Q.E.D.}$$

The following lemmas involve the "modified" order matrix concept, in that $\underset{\mathcal{A}}{>}$ and $\underset{\mathcal{B}}{>}$ may include relations not derivable by collinear comparison.

LEMMA 5. Let $\mathcal{A}$ be in general position; let $\overline{pq}$ be maximal in $\underset{\mathcal{A}}{>}$; and let the elements of $\mathcal{B}$ be the elements of $\mathcal{A}$ with column q deleted.** Then $C(\mathcal{B}) \subseteq C(\mathcal{A})$.

---

* Another $\dot{B}$, suggested by O. Gross, is given by

$$b_{ij} = a_{ij} - \Sigma_i \dot{x}_i a_{ij} - \Sigma_j \dot{y}_j a_{ij} + 2\ \Sigma_{ij} \dot{x}_i \dot{y}_j a_{ij},$$

with $(\dot{x}, \dot{y})$ all-positive but close to $(\dot{x}^o, \dot{y}^o)$.

** Contrast $\mathcal{B}$ with the $\mathcal{A}^q$ of Lemma 1.

PROOF. Take $\overline{rs} \in C(\mathcal{B})$. Using Lemma 3 and the fact that $\mathcal{A}$ is in general position, find $A \in \mathcal{A}$ such that the associated $A^q \in \mathcal{B}$ has a unique solution (x^*, y^*) with $x_r^* > 0$ and $y_s^* > 0$. Define the functions

$$f(x) = \min_{j \neq q} \sum_i a_{ij} x_i, \qquad g(y) = \max_i \sum_{j \neq q} a_{ij} y_j .$$

Since $f(x)$ is polyhedral (piecewise linear) and has a unique maximum at x^*, there exists $\alpha > 0$ such that

$$f(x) \leq \text{val}[A^q] - \alpha \|x^* - x\| \tag{3.1}$$

for all mixed-strategy vectors x. Similarly, there exists $\beta > 0$ such that

$$g(y) \geq \text{val}[A^q] + \beta \|y^* - y\| \tag{3.2}$$

for all mixed-strategy vectors y with $y_q = 0$.

Now let $A(\omega)$ be obtained from A by adding $\omega > 0$ to the element a_{pq}, and let $(x(\omega), y(\omega))$ be any solution of $A(\omega)$. Then certainly $\text{val}[A(\omega)] \leq f(x(\omega))$, and we have by (3.1)

$$\alpha \|x^* - x(\omega)\| \leq \text{val}[A^q] - \text{val}[A(\omega)].$$

By Lemma 4, $x(\omega) \to x^*$ as $\omega \to \infty$; hence $x_r(\omega)$ is positive for large ω.

The corresponding statement for $y_s(\omega)$ is a bit harder to prove. First we observe that

$$\sum_j a_{pj} y_j(\omega) + \omega y_q(\omega) \leq \text{val } A(\omega) \leq a,$$

where $a = \max_{ij} |a_{ij}|$. Hence

$$y_q(\omega) \leq 2a/\omega.$$

Then we have

$$\begin{aligned} \mathrm{val}[A(\omega)] &= \max_i \sum_j a_{ij}(\omega)y_j(\omega) \geq \max_i \sum_j a_{ij}y_j(\omega) \\ &= \max_i \left[\sum_{j\neq q} a_{ij}y_j(\omega) + a_{is}y_q(\omega) + (a_{iq} - a_{is})y_q(\omega)\right] \\ &\geq \max_i \left[\sum_{j\neq q} a_{ij}y_j(\omega) + a_{is}y_q(\omega)\right] - 2ay_q(\omega) \\ &\geq g(y'(\omega)) - 4a^2/\omega, \end{aligned}$$

where $y'(\omega)$ is like $y(\omega)$ except for $y'_q(\omega) = 0$ and $y'_s(\omega) = y_s(\omega) + y_q(\omega)$. Hence, by (3.2)

$$\mathrm{val}[A(\omega)] \geq \mathrm{val}[A^q] + \beta\|y^* - y'(\omega)\| - 4a^2/\omega,$$

and $y'(\omega)$ converges to y^*, by Lemma 4. Since $y_q(\omega) \to 0$, we have $y_s(\omega) \to y^*_s > 0$. Thus both r and s are active in $A(\omega)$ for large ω. But $A(\omega) \in \mathfrak{A}$ for all positive ω. Hence $\overline{rs} \in C(\mathfrak{A})$. Q.E.D.

LEMMA 6. If $\mathfrak{B}$ is obtained from $\mathfrak{A}$ by the deletion (as in Lemma 5) of a strictly majorized row or a strictly minorized column, then $C(\mathfrak{B}) \subseteq C(\mathfrak{A})$.

PROOF. A strictly dominated strategy can never be active.

THEOREM 3.4. If $\mathfrak{A}$ is in general position, then $R^{\#}(\mathfrak{A}) \subseteq C(\mathfrak{A})$.

PROOF. The theorem follows directly from Lemma 5 (and its "row" counterpart), Lemma 6, and the definition of restricted residual.

3.7. The Antisaddle

We close this part with a simple result concerning antisaddlepoints, i.e., saddlepoints of the negative matrix.

THEOREM 3.5. A strict antisaddlepoint of $\mathfrak{A}$ cannot be in the center, unless $\mathfrak{A}$ is 1-by-1.

PROOF. Suppose $S(-\mathcal{A}) = \{\overline{pq}\} \subseteq C(\mathcal{A})$. Take $A \in \mathcal{A}$ such that $\overline{pq} \in C(A)$, and let $\dot{A}$ be the submatrix of A defined by $C(A)$. Then $\dot{A}$ has an equalizer solution, which solves $-\dot{A}$ as well. But $-\dot{A}$ has a strict saddlepoint at $\overline{pq}$, and hence a unique, pure-strategy solution. Therefore $\dot{A}$ is 1-by-1 and $\overline{pq}$ is a saddlepoint of A as well as of $-A$, strict in both cases. This is possible only if A is 1-by-1. Q.E.D.

Weak antisaddlepoints are sometimes found in the center, as shown (starred) in the accompanying examples. The smaller matrix has a saddlepoint; the larger one (which is symmetric in the sense of § 1) does not.

1*	1
0	2

0*	0	0	−1
0	1	−1	1
0	−1	1	1
1	−1	−1	0

Similar reasoning can be used to establish the following more general result, which applies however only to numerical matrices:

THEOREM 3.6. If $S(A) \subseteq C(-A)$ then $S(A) = C(-A)$.

§ 4. INSTANTANEOUS GAMES

4.1. Games of Almost-perfect Information

There is a class of games of timing, typified by the so-called "noisy duels" ([5], p. 128-134; also [11], Vol. II), in which the payoff depends discontinuously upon the order in which certain actions occur, but is a continuous function of the occurrence times, given the sequence of events. The (finite) set of actions available to a player at any moment may depend on what has gone before; in any case the players are informed at all times of all all previous history. Except for the possibility of simultaneous action, a solution in pure strategies would be indicated. This possibility, however, not only makes mixed strategies necessary in general, but even affects the existence of a value. Indeed, there are some surprisingly simple indeterminate (valueless) games of this type, perhaps worthy of a place beside the examples of Ville ([5], p. 115) and of Sion and Wolfe ([18]). (See below, Fig. 1.)

Simultaneity is not a serious problem in the middle of the game, since either player can avoid it at negligible cost by altering his timing by a small random amount. But this tactic may not work at the beginning or end of the game, since the random displacement would be in a fixed direction.

In order to focus on the problem of determinateness, in games of timing with almost-perfect information, we shall assume that there is a "critical instant" at the beginning of the game. The players will be motivated (by the payoffs) to take some action either at $t = 0$ or immediately thereafter. We thereby reduce the infinite game in extensive form to an essentially finite, "instantaneous" game, whose payoffs are defined by the values of the subgames that result after the first action, or pair of simultaneous actions.

One of our results (Theorem 4.4) states that an instantaneous game has a value provided that the payoff in the event of simultaneous actions lies between the payoffs for the same actions performed singly. Another result (Theorem 4.5) shows how to assign a formal value to the game when the true value does not exist.

4.2. Some Indeterminate Noisy Duels

Two gunfighters are crouched behind barriers, where they can neither see nor be seen. If they stand up at the same instant, we may assume that their chances are equal if they both shoot at once, or if they both hold their fire. If only one fires, that one wins. If one stands up without shooting, and finds the other still down, then he can take control of the situation, and eventually win. But if the one under cover hears the other stand up and fire, then *he* can stand up and take control before his opponent has time to reload.

This (with apologies to the real world!) is an example of an indeterminate noisy duel. The critical moment is at the beginning of the time interval, since to wait an appreciable amount of time is definitely bad strategy. The situation at $t = 0$ can be represented by the following matrix:

	Stand up shooting	Stand up quiet	Wait
Stand up shooting ---	0	+1	−1
Stand up quiet ---	−1	0	+1
Wait -----	+1	−1	Ⓡ

Here "Ⓡ" is meant to suggest "repeat" or "replay," as in a recursive game (see § 4.4 below).

Let us test for a value. In the <u>minorant</u> game, in which player I openly commits himself to a particular (mixed) strategy, the outcome Ⓡ is worth no more than −1, because player II, knowing the probable duration of the other's "wait," will almost always succeed in acting first. Setting Ⓡ = −1 and solving, we find that the value is −1/9. By symmetry, the value of the <u>majorant</u> game is +1/9. Hence the duel is indeterminate.

An even simpler example is shown at right. The minorant value is zero; the majorant value is 1/3. Any matrix order-equivalent to this one (see § 3) also represents a game without a value, if we interpret "Ⓡ" as a number between 0 and 1 in determining order equivalence.

1	−1
−1	1
0	Ⓡ

The above can be viewed as a game on two unit squares, with discontinuities along the diagonals (Fig. 1).

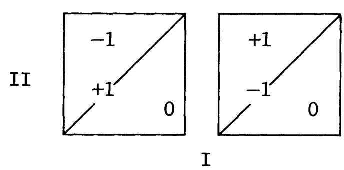

Figure 1.

4.3. The General Case

Let M(φ) denote the following m+1-by-n+1 array:

	"act"			"wait"
"act"	a_{11}	...	a_{1n}	a_{10}
	⋮		⋮	⋮
	a_{m1}		a_{mn}	a_{m0}
"wait"	a_{01}		a_{0n}	φ

Here φ is either a real number or the symbol Ⓡ. The game represented by M(Ⓡ) is said to be instantaneous if

$$\min_{j\neq 0} a_{0j} < \max_{i\neq 0} a_{i0}. \tag{4.1}$$

This condition puts a premium on immediate action. Without it, at least one player would be willing to delay, and other elements of the original game of timing (from which we assume the data in M(Ⓡ) to have been drawn) would become significant in the selection and timing of the first action. We shall be concerned henceforth only with the instantaneous case.

The minorant and majorant games have the matrices

$$M_1 = M(\min_{j\neq 0} a_{0j}) \quad \text{and} \quad M_2 = M(\max_{i\neq 0} a_{i0}),$$

respectively. Let their values be v_1 and v_2, and their sets of optimal strategies X_1 and X_2 (for player I) and Y_1 and Y_2 (for player II). Obviously,

$$\min_{j\neq 0} a_{0j} \leq v_1 \leq v_2 \leq \max_{i\neq 0} a_{i0}. \tag{4.2}$$

If $v_1 < v_2$ the game is indeterminate; if $v_1 = v_2$ the game is determinate and the optimal strategies for the two players correspond exactly to the elements of X_1 and Y_2, respectively.

THEOREM 4.1. A necessary and sufficient condition for the instantaneous game M(Ⓡ) to be determinate is that $X_1 \subseteq X_2$ and $Y_2 \subseteq Y_1$.

PROOF. Necessity. Any strategy guaranteeing v_1 to player I in M_1 will guarantee him at least as much in M_2. Therefore, $v_1 = v_2$ implies $X_1 \subseteq X_2$. Similarly, $Y_2 \subseteq Y_1$.

Sufficiency. Let $X_1 \subseteq X_2$, $Y_2 \subseteq Y_1$, and suppose $v_1 < v_2$. Take $x \in X_1$. Then, since x is optimal in M_2, we have

$$\sum_{i=0}^{m} x_i a_{ij} \geq v_2 > v_1, \quad j = 1, \ldots, n.$$

Hence no $j \neq 0$ can be active in M_1. Hence Y_1 consists of the single strategy "wait." The same holds for Y_2, by inclusion, and for X_2 and X_1, by symmetry. Hence M_1 and M_2 have saddlepoints at "wait-wait," contradicting (4.1). Q.E.D.

THEOREM 4.2. A necessary condition for the instantaneous game M(Ⓡ) to be determinate is that "wait" is unplayable (i.e., not active) for at least one player in at least one of M_1, M_2.

PROOF. Let $v_1 = v_2$, and take $x \in X_1$, $y \in Y_2$. Then $x \in X_2$ and $y \in Y_1$, by Theorem 4.1, and we have

$$0 = v_2 - v_1 = xM_2y - xM_1y = x_0y_0(\max_{i \neq 0} a_{i0} - \min_{j \neq 0} a_{0j}).$$

By (4.1), $x_0y_0 = 0$. Since x and y may be chosen independently, it follows that either x_0 vanishes identically for $x \in X_1$, or y_0 vanishes identically for $y \in Y_2$. Q.E.D.

THEOREM 4.3. A sufficient condition for the instantaneous game M(Ⓡ) to be determinate is that "wait" is unplayable for both players in at least one of M_1, M_2.

PROOF. Under the hypothesis, val[M(a)] is independent of a.

The conditions given so far are somewhat impractical, since the values of the games M_1, M_2 are at least as easy to find as the optimal strategies. The condition in the next theorem is free from this drawback, and it also has a simple heuristic interpretation, namely: if two actions occur simultaneously, the result is intermediate in value between the results of either action performed separately.

THEOREM 4.4. If (4.1) holds, and, if, for all $i \neq 0$, $j \neq 0$.

$$(4.3) \qquad \min(a_{i0}, a_{0j}) \leq a_{ij} \leq \max(a_{i0}, a_{0j}),$$

then $M(Ⓡ)$ is determinate.

PROOF. Let i^*, j^* be such that $a_{i^*0} = \max_{i \neq 0} a_{i0}$, $a_{0j^*} = \min_{j \neq 0} a_{0j}$. Let I be the set of $i \neq 0$ such that $a_{i0} \geq a_{0j^*}$, and J the set of $j \neq 0$ such that $a_{0j} \leq a_{i^*0}$. Then by (4.1), $i^* \in I$ and $j^* \in J$, so that neither I nor J nor their complements are empty. Take $j \in J$ and $i \in I$. If $i \neq 0$ then we have

$$a_{i0} < a_{0j^*} \leq a_{0j} \leq a_{i^*0},$$

by the definitions of I, j^*, and J, respectively. Applying the "betweenness" condition (4.3) twice, we obtain

$$a_{ij} \leq a_{0j} \leq a_{i^*j}.$$

Thus, for every $i \notin I$ (including $i = 0$), $a_{ij} \leq a_{i^*j}$ holds for all $j \in J$. Similarly, for every $j \in J$, $a_{ij} \geq a_{ij^*}$ holds for all $i \in I$. This means that $I \times J$ is a weak generalized saddlepoint of $M(a)$ (see § 3.1). Hence $\text{val}[M(a)]$ is independent of a. Q.E.D.

4.4. A Way of Resolving the Indeterminacy

One may wish to assign a formal value to the instantaneous game $M(Ⓡ)$ when no true value exists. A way to do this — perhaps the only "fair" way — is to pursue the analogy with stochastic or recursive games [16], [6], and attempt to solve the equation

$$(4.4) \qquad a = \text{val}[M(a)].$$

In effect, we replace the time continuum by a discrete, well-ordered set of points, making simultaneity after $t = 0$ a significant possibility. We shall see that condition (4.1), which characterizes "instantaneous" games, is intimately related to the existence of a unique solution to (4.4).

THEOREM 4.5. If $M(Ⓡ)$ is an instantaneous game, then (4.4) has a unique solution $a = \bar{v}$, and we have $v_1 \leq \bar{v} \leq v_2$.

PROOF. Uniqueness. Let (4.4) have two solutions $\bar{v} < \bar{\bar{v}}$. Let x be optimal in $M(\bar{\bar{v}})$ for player I and let y be optimal in $M(\bar{v})$ for player II. Then $xM(\bar{\bar{v}})y \geq \bar{\bar{v}}$ and $xM(\bar{v})y \leq \bar{v}$. Subtracting, we obtain

$$\bar{\bar{v}} - \bar{v} \leq x(M(\bar{\bar{v}}) - M(\bar{v}))y = x_0 y_0 (\bar{\bar{v}} - \bar{v}).$$

Hence $x_0 = y_0 = 1$, and x and y are pure. Thus, for all $i \neq 0$, $j \neq 0$,

$$a_{0j} \geq \bar{\bar{v}} > \bar{v} \geq a_{i0}.$$

This contradicts (4.1).

Existence. The function $F(a) = \text{val}[M(a)]$ is monotonic nondecreasing. Hence, by (4.2), we have

$$F(\min_{j \neq 0} a_{0j}) \leq F(v_1);$$

in other words $v_1 - F(v_1) \leq 0$. Similarly, $v_2 - F(v_2) \geq 0$. But $a - F(a)$ is continuous; hence it has a zero in $[v_1, v_2]$. Q.E.D.

COROLLARY. In the determinate case, $\bar{v}$ is equal to the true value.

We note that if (4.1) is not satisfied, then all values in the interval $[\max a_{i0}, \min a_{0j}]$ are solutions of (4.4), and no others.

From the fact that $F(a)$ and $a - F(a)$ are both monotonic nondecreasing functions, it follows that the sequence $b, F(b), F(F(b)), \ldots$ (b arbitrary) converges monotonically to a solution of (4.4). This can be useful in making sharper numerical estimates; e.g., we have $F(v_1) \leq \bar{v} \leq F(v_2)$, etc.

In the first indeterminate game of § 4.2 we have $\bar{v} = 0$, by symmetry. In the second example we also have $\bar{v} = 0$; thus, the first player's advantage in this game, such as it is, vanishes if time is made discrete. This feature, also present in the order-equivalent variants, shows that strict inequality need not hold in Theorem 4.5 for indeterminate games.

§ 5. FICTITIOUS PLAY IN NON-ZERO-SUM GAMES

5.1. Discussion

The method of fictitious play (FP) resembles a multistage learning process. At each stage, it is assumed that the players choose a strategy that would yield the optimum result if employed against all past choices of their opponents. Various conventions can be adopted with regard to the first move, indifferent alternatives, simultaneous vs. alternating moves, and weighting of past choices. The method can meaningfully be applied to any finite game, and to many infinite games as well. (See [3], [4], [5] pp. 82-85, and [14].)

It was once conjectured that the mixed strategies defined by the accumulated choices of the players would always converge to the equilibrium point of the game, or, in the event of nonuniqueness, to a set of mutually compatible equilibrium points. This is the natural generalization of Robinson's theorem [14] for the zero-sum two-person case; it was recently verified by Miyasawa [12] for the special case of two players with two pure strategies apiece.

The trouble begins, as we shall see, as soon as we add a third strategy for each player. It appears, intuitively, that this size is necessary to produce enough variety; if FP is to fail, the game must contain elements of both coordination and competition. Our counterexamples include a whole class of order-equivalent games, and thus do not depend on numerical quirks in the payoff matrices; nor are they sensitive to the minor technicalities of the FP algorithm. It is clear that games with more players, or with more strategies per player, can exhibit the same kind of misbehavior.

5.2. A Class of Nonconvergent Examples

We shall elaborate slightly on the game described in the Introduction, to eliminate any question of "degeneracy." The payoff matrices are shown at right. We assume that $a_i > b_i > c_i$ and $\alpha_i > \beta_i > \gamma_i$, for $i = 1, 2, 3$. It follows that the game is not constant-sum. It is easily shown that the equilibrium point must be completely mixed (all strategies active), and hence unique.

$$\underset{\text{I}}{\begin{bmatrix} a_1 & c_2 & b_3 \\ b_1 & a_2 & c_3 \\ c_1 & b_2 & a_3 \end{bmatrix}} \quad \underset{\text{II}}{\begin{bmatrix} \beta_1 & \gamma_1 & \alpha_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \gamma_3 & \alpha_3 & \beta_3 \end{bmatrix}}$$

For simplicity, we shall assume that the FP choices are made simultaneously, and that the first choice pair is $\overline{11}$. Consider any occurrence of $\overline{11}$ in the FP sequence. The next choice of player I will certainly be 1 again, since that strategy will have become more desirable. Player II will either stay with 1, or shift to 3. Eventually, in fact, he must shift to 3, since $\alpha_1 > \beta_1$. Thus, after each run of $\overline{11}$ we will find a run of $\overline{13}$. By a similar argument, this will be followed by a run of $\overline{33}$, and then runs of $\overline{32}$, $\overline{22}$, $\overline{21}$, $\overline{11}$, ..., in a never-ending cycle.

Suppose that a run of $\overline{11}$ is just about to begin. Let X represent the current "accumulated-choices" vector for player I (thus, X_i is the number of times he has chosen row i), and let Y be the same for player II. Let H denote the current "comparative-payoffs" vector for player I (thus, $H_1 = Y_1a_1 + Y_2c_2 + Y_3b_3$, etc.). Since he is about to choose 1, we have

$$H_1 = \max(H_1, H_2, H_3).$$

Let there occur r_{11} choices of $\overline{11}$, followed by r_{13} choices of $\overline{13}$. Then I's new comparative payoffs are

$$\begin{cases} H_1' = H_1 + r_{11}a_1 + r_{13}b_3, \\ H_2' = H_2 + r_{11}b_1 + r_{13}c_3, \\ H_3' = H_3 + r_{11}c_1 + r_{13}a_3. \end{cases}$$

Since he shifts to 3 at this point, we must have $H_3' \geq H_1'$. But $H_3 \leq H_1$; hence $H_3' - H_3 \geq H_1' - H_1$, and we have

$$r_{13} \geq \frac{a_1 - c_1}{a_3 - b_3} r_{11}.$$

Let r_{33} be the length of the $\overline{33}$ run that follows. By analogous reasoning, we have

$$r_{33} \geq \frac{\alpha_1 - \gamma_1}{\alpha_3 - \beta_3} r_{13}.$$

Repeating the argument four times more, we obtain

$$r_{11}' \geq \left(\prod_{i=1}^{3} \frac{(a_i - c_i)(\alpha_i - \gamma_i)}{(a_i - b_i)(\alpha_i - \beta_i)} \right) r_{11}, \tag{5.1}$$

where r_{11}' denotes the length of the next $\overline{11}$ run. Since the constant in (5.1) is greater than 1, the runs of $\overline{11}$ increase in length exponentially (or faster). The same rate of increase occurs for the other choice pairs. Hence the ratios X_i/X_j, Y_i/Y_j, $i \neq j$, do not converge. In particular, the normalized strategy vectors $x = X/\Sigma X_i$, $y = Y/\Sigma Y_i$ do not converge to the unique equilibirum point strategies.

It can be shown that x and y, in their respective strategy spaces, approach limit cycles which do not contain the equilibrium point strategies. Hence there is no convergence to an equilibrium point even through subsequences.

The argument we have given is independent of the tie-breaking rule. With minor modifications it can also handle the case of alternating moves, as well as the case of nonintegral run lengths. The latter implies that the differential-equation version of FP (see [3]) will also fail to converge to the solution.

5.3. An Example

We have computed the limit cycles for the numerical example shown at right. The payoffs do not satisfy the strict inequalities assumed in the preceding proof, and the constant in the estimate (5.1) is exactly 1. A more refined analysis, however, shows that the run lengths do increase exponentially; in fact, the ratio of r_{11}' to r_{11} tends in the limit to

1	0	0		0	0	1
0	1	0		1	0	0
0	0	1		0	1	0
	I				II	

θ^6, where $\theta \doteq 1.466$ is a root of $\theta^3 - \theta^2 = 1$.

The limit cycles in the separate strategy spaces are illustrated in Fig. 2. In the product space they form a hexagon; typical vertices are

$$(\bullet) \quad (1,\ \theta^4,\ \theta^2;\ \theta^3,\ \theta^3,\ \theta)/C,$$

$$(\blacksquare) \quad (\theta^4,\ \theta^4,\ \theta^2;\ \theta^5,\ \theta^3,\ \theta)/C\theta, \text{ etc.},$$

where C is a normalizing constant. The unique equilibrium point is

$$(\odot) \quad (1,\ 1,\ 1;\ 1,\ 1,\ 1)/3.$$

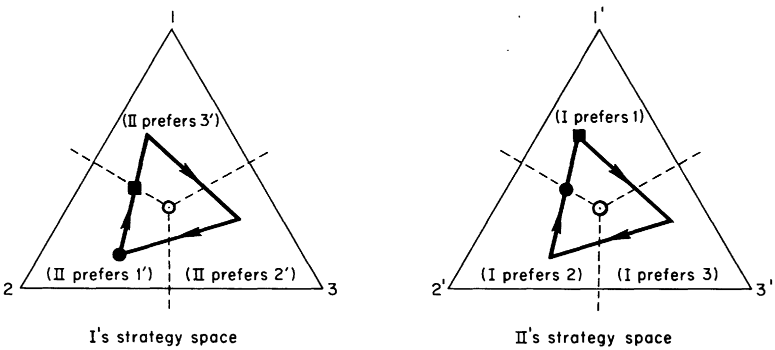

Figure 2

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