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The general graph matching game: Approximate core

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ABSTRACT

The classic paper of Shapley and Shubik (1971) characterized the core of the assignment game using ideas from matching theory and LP-duality theory and their highly non-trivial interplay. Whereas the core of this game is always non-empty, that of the general graph matching game can be empty.

This paper salvages the situation by giving an imputation in the 2/3-approximate core for the latter; moreover this imputation can be computed in polynomial time. This bound is best possible, since it is the integrality gap of the natural underlying LP. Our profit allocation method goes further: the multiplier on the profit of an agent is often better than $\frac{2}{3}$ and lies in the interval $[\frac{2}{3}, 1]$, depending on how severely constrained the agent is.

1. Introduction

The matching game forms one of the cornerstones of cooperative game theory. This game can also be viewed as a matching market in which utilities of the agents are stated in monetary terms and side payments are allowed, i.e., it is a *transferable utility (TU) market*. A quintessential solution concept in this theory is the *core*, which captures all possible ways of distributing the total worth of a game among individual agents in such a way that the grand coalition remains intact, i.e., a sub-coalition will not be able to generate more profits by itself and therefore has no incentive to secede from the grand coalition. An imputation in the core can also be viewed as a “fair” way of distributing the worth of a game among individual agents. Additionally, the core also provides profound insights into the negotiating power of individuals and sub-coalitions, e.g., see Section 3.1. For an extensive coverage of these notions, see the book by Moulin (2014).

The special case of the matching game when the underlying graph is bipartite is called the *assignment game*. The classic paper of Shapley and Shubik (1971) characterized profit-sharing methods that lie in the core of such games by using ideas from matching theory and LP-duality theory and their highly non-trivial interplay; in particular, the core is always non-empty.

On the other hand, for games defined over general graphs, the core is not guaranteed to be non-empty, see Section 2 for an easy proof. The purpose of this paper is to salvage the situation to the extent possible by giving a notion of approximate core for such games. Our proposal is different from the notions given in game theory to deal with the emptiness of core; these include the least core and nucleolus. As detailed in Section 3, whereas these notions involve additive approximation,

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our method involves multiplicative approximation. In Section 3 we argue that the allocations yielded by the latter are more fair. Section 3.1 shows that an imputation in the 2/3-approximate core is fair even to unmatched agents and it minimizes, among all profit sharing methods, the worst case percentage gain which a seceding sub-coalition can achieve.

The approximation factor we achieve is 2/3. This is best possible, since it is the integrality gap of the underlying LP; this follows easily from an old result of Balinski (1965) characterizing the vertices of the polytope defined by the constraints of this LP. An interesting feature of our profit-sharing mechanism is that it restricts only the most severely constrained agents to a multiplier of 2/3, and the less severely constrained an agent is, the better is her multiplier, all the way to 1; bipartite graphs belong to the last category. One way of stating the improved factor is: if the underlying graph has no odd cycles of length less than $2k+1$, then our factor is $\frac{2k}{2k+1}$. For all the reasons stated above, our proposal represents a valuable normative criterion for distributing the profit of a game when the core is empty.

The following setting, taken from Eriksson and Karlander (2001) and Biró et al. (2012), vividly captures the underlying issues. Suppose a tennis club has a set V of players who can play in an upcoming doubles tournament. Let $G=(V,E)$ be a graph whose vertices are the players and an edge (i,j) represents the fact that players i and j are compatible doubles partners. Let w be an edge-weight function for G, where w_{ij} represents the expected earnings if i and j do partner in the tournament. Then the total worth of agents in V is the weight of a maximum weight matching in G. Assume that the club picks such a matching M for the tournament. The question is how to distribute the total profit among the agents — strong players, weak players and unmatched players — so that no subset of players feel they will be better off seceding and forming their own tennis club.

2. Definitions and preliminary facts

Definition 1. The *general graph matching game* consists of an undirected graph $G=(V,E)$ and an edge-weight function w. The vertices $i \in V$ are the agents and an edge (i,j) represents the fact that agents i and j are eligible for an activity, for concreteness, let us say that they are eligible to participate as a doubles team in a tournament. If $(i,j) \in E$, w_{ij} represents the profit generated if i and j play in the tournament.

Definition 2. The *worth* of a coalition $S \subseteq V$ is defined to be the maximum profit that can be generated by teams within S and is denoted by $p(S)$. Formally, $p(S)$ is the weight of a maximum weight matching in the graph G restricted to vertices in S only. The *worth of the game* is defined to be $p(V)$, i.e., the worth of the *grand coalition, V*. The *characteristic function* of the game is defined to be $p: 2^V \rightarrow \mathcal{R}_+$.

Definition 3. An *imputation*[2] gives a way of dividing the worth of the game, $p(V)$, among the agents. Formally, it is a function $v: V \rightarrow \mathcal{R}_+$ such that $\sum_{i \in V} v(i) = p(V)$.

Definition 4. An imputation v is said to be in the *core of the matching game* if for any coalition $S \subseteq V$, the total worth allocated to agents in S is at least as large as the worth that they can generate by themselves, i.e., $v(S) \geq p(S)$, where $v(S) = \sum_{i \in S} v(i)$.

We next describe the characterization of the core of the assignment game given by Shapley and Shubik (1971). Shapley and Shubik had described this game in the context of the housing market in which agents are of two types, buyers and sellers. They had shown that each imputation in the core of this game gives rise to unique prices for all the houses. In this paper we will present the assignment game in a variant of the tennis setting given in the Introduction; this will obviate the need to define "prices", hence leading to simplicity.

Suppose a coed tennis club has sets U and V of women and men players, respectively, who can participate in an upcoming mixed doubles tournament. Assume $|U|=m$ and $|V|=n$, where m, n are arbitrary. Let $G=(U,V,E)$ be a bipartite graph whose vertices are the women and men players and an edge (i,j) represents the fact that agents $i \in U$ and $j \in V$ are eligible to participate as a mixed doubles team in the tournament. Let w be an edge-weight function for G, where w_{ij} represents the expected earnings if i and j do participate as a team in the tournament. We will assume that $w(i,j) > 0$; this is reasonable since i and j are compatible. Once again, the total worth of the game is the weight of a maximum weight matching in G.

The linear program (1) gives the LP-relaxation of the problem of finding such a matching. In this program, variable x_{ij} indicates the extent to which edge (i,j) is picked in the solution. Matching theory tells us that this LP always has an integral optimal solution Lovász and Plummer (1986); the latter is a maximum weight matching in G.

[2] Some authors prefer to call this a pre-imputation, while using the term imputation when individual rationality is also satisfied.

$$\begin{aligned}
\max \quad & \sum_{(i,j)\in E} w_{ij}x_{ij} \\
\text{s.t.} \quad & \sum_{(i,j)\in E} x_{ij} \le 1 \quad \forall i \in U, \\
& \sum_{(i,j)\in E} x_{ij} \le 1 \quad \forall j \in V, \\
& x_{ij} \ge 0 \quad \forall (i,j) \in E
\end{aligned} \tag{1}$$

Taking u_i and v_j to be the dual variables for the first and second constraints of (1), we obtain the dual LP:

$$\begin{aligned}
\min \quad & \sum_{i\in U} u_i + \sum_{j\in V} v_j \\
\text{s.t.} \quad & u_i + v_j \ge w_{ij} \quad \forall (i,j) \in E, \\
& u_i \ge 0 \quad \forall i \in U, \\
& v_j \ge 0 \quad \forall j \in V
\end{aligned} \tag{2}$$

For the assignment game, the definition of an imputation, Definition 3, needs to be modified in an obvious way to distinguish the profit shares of women and men players. We will denote an imputation by (u, v).

Theorem 1. *(Shapley and Shubik (1971)) The imputation (u, v) is in the core of the assignment game if and only if it is an optimal solution to the dual LP, (2).*

For general graphs, the LP relaxation of the maximum weight matching problem is an enhancement of that for bipartite graphs via odd set constraints, as given below in (3). The latter constraints are exponential in number, namely for every odd subset S of vertices. Clearly, the total number of integral matched edges in this set can be at most $\frac{(|S|-1)}{2}$. The constraint imposes this bound on any fractional matching as well, thereby ensuring that a fractional matching that picks each edge of an odd cycle half-integrally is disallowed and there is always an integral optimal matching.

$$\begin{aligned}
\max \quad & \sum_{(i,j)\in E} w_{ij}x_{ij} \\
\text{s.t.} \quad & \sum_{(i,j)\in E} x_{ij} \le 1 \quad && \forall i \in V, \\
& \sum_{(i,j)\in S} x_{ij} \le \frac{(|S|-1)}{2} \quad && \forall S \subseteq V,\ S \text{ odd}, \\
& x_{ij} \ge 0 \quad && \forall (i,j) \in E
\end{aligned} \tag{3}$$

The dual of this LP has, in addition to variables corresponding to vertices, v_i, exponentially many more variables corresponding to odd sets, z_S, as given in (4). As a result, the entire worth of the game does not reside on vertices only — it also resides on odd sets.

$$\begin{aligned}
\min \quad & \sum_{i\in V} v_i + \sum_{S\subseteq V,\ \text{odd}} z_S \\
\text{s.t.} \quad & v_i + v_j + \sum_{S\ni i,j} z_S \ge w_{ij} \quad && \forall (i,j) \in E, \\
& v_i \ge 0 \quad && \forall i \in V, \\
& z_S \ge 0 \quad && \forall S \subseteq V,\ S \text{ odd}
\end{aligned} \tag{4}$$

There is no natural way of dividing z_S among the vertices in S to restore core properties. The situation is more serious than that: it turns out that in general, the core of a non-bipartite game may be empty.

A (folklore) proof of the last fact goes as follows: Consider the graph K_3, i.e., a clique on three vertices, i, j, k, with a weight of 1 on each edge. Any maximum matching in K_3 has only one edge, and therefore the worth of this game is 1. Suppose there is an imputation v which lies in the core. Consider all three two-agent coalitions. Then, we must have:

$$v(i) + v(j) \ge 1, \quad v(j) + v(k) \ge 1 \quad \text{and} \quad v(i) + v(k) \ge 1.$$

This implies $v(i) + v(j) + v(k) \geq 3/2$ which exceeds the worth of the game, giving a contradiction.

The following definition is central to our paper; in particular, it will be used to get around the fact that the core of the game corresponding to the graph K_3 is empty.

Definition 5. Let $p : 2^V \to \mathcal{R}_+$ be the characteristic function of a game and let $1 \geq \alpha > 0$. An imputation $t : V \to \mathcal{R}_+$ is said to be in the *α-approximate core* of the game if:

1. The total profit allocated by t is at most the worth of the game, i.e.,

$$\sum_{i \in V} t_i \leq p(V).$$

2. The total profit accrued by agents in a sub-coalition $S \subseteq V$ is at least α fraction of the profit which S can generate by itself, i.e.,

$$\forall S \subseteq V : \sum_{i \in S} t_i \geq \alpha \cdot p(S).$$

If imputation t is in the α-approximate core of a game, then the ratio of the total profit of any sub-coalition on seceding from the grand coalition to its profit while in the grand coalition is bounded by a factor of at most $\frac{1}{\alpha}$.

Consider the graph K_3 again. Observe that if we distribute the worth of this game as follows, we get a 2/3-approximate core allocation: $v(i) = v(j) = v(k) = 1/3$. Now each edge is covered to the extent of 2/3 of its weight. In Section 4 we show that such an approximate core allocation can always be obtained for the general graph matching game. In Section 4 we will also need the following notion.

Definition 6. Consider a linear programming relaxation for a maximization problem. For an instance I of the problem, let $\text{OPT}(I)$ denote the weight of an optimal integral solution to I and let $\text{OPT}_f(I)$ denote the weight of an optimal fractional solution to I, i.e., the solution given by the LP-relaxation of I. Then, the *integrality gap* of this LP-relaxation is defined to be:

$$\inf_I \frac{\text{OPT}(I)}{\text{OPT}_f(I)}.$$

3. Comparison with other solution concepts

The core is a key solution concept in cooperative game theory for several reasons. First, imputations in the core give a way of distributing the worth of the game among players so the grand coalition remains intact, i.e., no sub-coalition can do better by itself and hence has any incentive to secede. Second, this way of distributing the worth can be considered to be *fair*, since each agent gets as much as her worth to the grand coalition and all sub-coalitions she is in.

An imputation in the core has to ensure that *each* of the exponentially many sub-coalitions is "happy" — clearly, that is a lot of constraints. As a result, the core is non-empty only for a handful of games, those with very good structural properties. Besides the assignment game, which is described in Section 2, this holds for the stable matching solution concept, given by Gale and Shapley (1962), which lies in the core of their game. The only coalitions that matter in this game are ones formed by one agent from each side of the bipartition; a stable matching ensures that no such coalition has the incentive to secede.

To deal with the question of emptiness of core, the following two notions have been given. The first is that of *least core*, defined by Maschler et al. (1979). If the core is empty, there will necessarily be sets $S \subseteq V$ such that $v(S) < p(S)$ for any imputation v. The least core maximizes the minimum of $v(S) - p(S)$ over all sets $S \subseteq V$, subject to $v(\emptyset) = 0$ and $v(V) = p(V)$. This involves solving an LP with exponentially many constraints, though, if a separation oracle can be implemented in polynomial time, then the ellipsoid algorithm will accomplish this in polynomial time Grotschel et al. (1988); see below for a resolution for the case of the matching game.

A more well known notion is that of *nucleolus* which is contained in the least core. After maximizing the minimum of $v(S) - p(S)$ over all sets $S \subseteq V$, it does the same for all remaining sets and so on. A formal definition is given below.

Definition 7. For an imputation $v : V \to \mathcal{R}_+$, let $\theta(v)$ be the vector obtained by sorting the $2^{|V|} - 2$ values $v(S) - p(S)$ for each $\emptyset \subset S \subset V$ in non-decreasing order. Then the unique imputation, v, that lexicographically maximizes $\theta(v)$ is called the *nucleolus* and is denoted $\nu(G)$.

The nucleolus was defined in 1969 by Schmeidler (1969), though its history can be traced back to the Babylonian Talmud Aumann and Maschler (1985). It has several modern-day applications, e.g., Brânzei et al. (2005). Clearly, every game has a nucleolus, even if its core is empty, and it provides valuable information. In 1998, Faigle et al. (1998) stated the problem of computing the nucleolus of the matching game in polynomial time. For the assignment game with unit weight edges, this was done in Solymosi and Raghavan (1994); however, since the assignment game has a non-empty core, this result was of little value. For the general graph matching game with unit weight edges, this was done by Kern and Paulusma (2003). Finally, the general problem was resolved by Könemann et al. (2020). However, their algorithm makes extensive use of the ellipsoid algorithm and is therefore neither efficient nor does it give deep insights into the underlying combinatorial structure. They leave the open problem of finding a combinatorial polynomial time algorithm.

The basic difference between the two notions defined above and our notion of α-approximate core is that whereas the former provide an additive approximation to core, we provide a multiplicative approximation. We would like to argue that the latter makes more sense: As an extreme example, consider the case that the worst set for least core, say S, has small $p(S)$. Then the members of this set are unfairly treated, since they may end up getting almost no part of the worth of the game. On the other hand, under multiplicative approximation, the profit of *every* coalition is guaranteed to be at least a fixed fraction of its worth, e.g., $\frac{2}{3} \cdot p(S)$ in our solution. Thus, an α-approximate core imputation is attempting to be fair to the extent possible, given that the core is empty.

There are other important differences as well. The difference $v(S) - p(S)$ appearing in the least core and nucleolus has not been upper-bounded for any standard family of games, including the general graph matching game. Moreover, since these notions involve the solution of LPs having an exponential number of constraints, polynomial time solvability is hard to establish and has been done only for the matching game. But even there, the algorithm is far from satisfactory, for reasons given above.

In contrast, our notion is steeped in the sophisticated methodology developed in the by-now mature field of approximation algorithms, e.g., see Vazirani (2001), which uses multiplicative approximation as a norm, and yields polynomial time algorithms. Therefore, we feel it is a simpler, more direct and more effective way of arriving at a "fair" division of the worth of a game in the face of an empty core.

3.1. *Fairness in the presence of unmatched agents*

One situation in which the fairness of a solution concept is particularly easy to discern is when some agents need to be necessarily left out, i.e., they are not involved in a trade, or a doubles team participating in an upcoming tournament. In the matching game, such situations arise when the maximum weight matching in the underlying graph is not a perfect matching, i.e., it leaves some vertices unmatched. Clearly K_3 is such an example. Let us first see how the core deals with such situations in an assignment game before discussing K_3.

Consider a bipartite graph having two edges, $(u, v_1), (u, v_2)$ on the three vertices u, v_1, v_2. Clearly, one of v_1 and v_2 will be left out in any matching. First assume that the weight of both edges is 100. If so, the unique imputation in the core gives zero to v_1 and v_2, and 100 to u. Next assume that the weights of the two edges are 100 and 101, respectively. If so, the unique imputation in the core gives $0, 1, 100$ to v_1, v_2 and u, respectively.

How fair are these imputations? As stated in the Introduction, imputations in the core have a lot to do with the negotiating power of individuals and sub-coalitions. Let us argue that when the imputations given above are viewed from this angle, they are fair in that the profit allocated to an agent is consistent with their negotiating power, i.e., their worth. In the first case, whereas u has alternatives, v_1 and v_2 don't. As a result, u will squeeze out all profits from whoever she plays with, by threatening to partner with the other player. Therefore v_1 and v_2 have to be content with no rewards! In the second case, u can always threaten to match up with v_2. Therefore v_1 has to be content with a profit of 1 only.

Let us study this phenomenon in a general assignment game $G = (U, V, E)$ with weight function $w : E \to \mathbb{Q}_+$. Let us denote a generic agent in $U \cup V$ by q. Let y_q be the allocation made to q by an imputation in the core of this game. We will say that q is *never paid* if $y_q = 0$ under *every* imputation in the core. We will say that q is *essential* if q is matched in every maximum weight matching in G and it is *not essential* otherwise. Using complementary slackness conditions for LPs (1) and (2), one can prove the following; for a proof see Theorem 2 in Vazirani (2022).

$$q \text{ is never paid} \iff q \text{ is not essential}$$

Thus the assignment game rewards only those agents who always play. This raises the following question: Can't a non-essential player, say q, team up with another player, say p, and secede, by promising p almost all of the resulting profit? The answer is "No", because the dual (2) has the constraint $y_q + y_p \geq w_{qp}$. Therefore, if $y_q = 0$, $y_p \geq w_{qp}$, i.e., p will not gain by seceding together with q.

The analogous questions have very different answers in the case of the general graph matching game, and their answers will provide valuable insights into our notion of 2/3-approximate core. Clearly, in K_3 under unit cost edges, all three teams are equally good and so are all three players. Therefore, fairness posits that their profits should be equal. Additionally, no matter which matching we pick, one of the players will be left unmatched. Therefore no player is essential. By the bipartite rule, all three players should get zero profit, which clearly makes little sense.

Next, suppose the three players are u, v, w and we match (u, v), splitting the profit between them in some manner and giving zero profit to w. Then w can team up with the agent getting smaller profit, say u, and together they can generate at least $1/2$ more, thereby giving u incentive to secede. On the other hand the unique imputation in the 2/3-approximate core allocates $1/3$ to each player. First, it treats the players fairly. Second, if u secedes with w, then they can generate only $1/3$ more. Clearly, there is no way of distributing the worth of the game so that the seceding team generates less.

Finally, let us introduce asymmetry in the example by changing the weight of (u, v) to $1 + \epsilon$. Then the instance has a unique maximum weight matching, namely match (u, v). Now the unique imputation in the 2/3-approximate core gives

$$\frac{1+\epsilon}{3}, \frac{1+\epsilon}{3}, \frac{1-\epsilon}{3}$$

to u, v, w, respectively. If u secedes with w, then they still generate $1/3$ more.

As per Definition 5, let t be an imputation in the 2/3-approximate core of a general graph matching game. Assume that sub-coalition $S \subset V$ secedes. By the definition, $\sum_{i \in S} t_i \geq \frac{2}{3} \cdot p(S)$. Therefore, the extra worth that S can generate on seceding is at most $\frac{1}{3} \cdot p(S)$, i.e., at most a factor of $1/3$ more. Therefore, K_3 is a tight example for the percentage gain achievable by seceding and as argued above, no other profit sharing method can do better.

Unlike a core imputation for the assignment game, which distributes the worth created by a matched edge only among the two matched agents, our notion of 2/3-approximate core makes transfers from the set of matched agents to the set of unmatched agents, thereby using more thoroughly the TU aspect of the general graph matching game. Furthermore, an imputation in the 2/3-approximate core is fair even to unmatched agents and it minimizes, among all profit sharing methods, the worst case percentage gain which a seceding sub-coalition can achieve.

4. A 2/3-approximate core for the matching game

We will work with the following LP-relaxation of the maximum weight matching problem, (5). This relaxation always has an integral optimal solution in case G is bipartite, but not in general graphs. In the latter, its optimal solution is a maximum weight fractional matching in G.

$$\begin{aligned} \max \quad & \sum_{(i,j) \in E} w_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{(i,j) \in E} x_{ij} \leq 1 \quad \forall i \in V, \\ & x_{ij} \geq 0 \quad \forall (i,j) \in E \end{aligned} \tag{5}$$

Taking v_i to be dual variables for the first constraint of (5), we obtain LP (6). Any feasible solution to this LP is called a *cover* of G since for each edge (i, j), v_i and v_j cover edge (i, j) in the sense that $v_i + v_j \geq w_{ij}$. An optimal solution to this LP is a *minimum cover*. We will say that v_i is the *profit* of vertex i.

$$\begin{aligned} \min \quad & \sum_{i \in V} v_i \\ \text{s.t.} \quad & v_i + v_j \geq w_{ij} \quad \forall (i,j) \in E, \\ & v_i \geq 0 \qquad \forall i \in V \end{aligned} \tag{6}$$

By the LP Duality Theorem, the weight of a maximum weight fractional matching equals the total profit of a minimum cover. If for graph G, LP (5) has an integral optimal solution, then it is easy to see that an optimal dual solution gives a way of allocating the total worth which lies in the core. Deng et al. (1997) prove that the core of this game is non-empty if and only if LP (5) has an integral optimal solution.

We will say that a solution x to LP (5) is *half-integral* if for each edge (i, j), x_{ij} is 0, 1/2 or 1. Balinski (1965) showed that the vertices of the polytope defined by the constraints of LP (5) are half-integral, see Theorem 2.

Theorem 2. *(Balinski (1965)) The vertices of the polytope defined by the constraints of LP (5) are half-integral.*

As a consequence of Theorem 2, LP (5) always has a half-integral optimal solution. Biró et al. (2012) gave an efficient algorithm for finding an optimal half-integral matching by using an idea of Nemhauser and Trotter (1975) of doubling edges, hence obtaining an efficient algorithm for determining if the core of the game is non-empty.

Our mechanism starts by using the doubling idea of Nemhauser and Trotter (1975). Transform $G = (V, E)$ with edge-weights w to graph $G' = (V', E')$ and edge weights w' as follows. Corresponding to each $i \in V$, V' has vertices i' and i'', and corresponding to each edge $(i, j) \in E$, E' has edges (i', j'') and (i'', j') each having a weight of $w_{ij}/2$.

Since each cycle of length k in G is transformed to a cycle of length $2k$ in G', the latter graph has only even length cycles and is bipartite. A maximum weight matching and a minimum cover for G' can be computed in polynomial time Lovász and Plummer (1986), say x' and v', respectively. Next, let

$$x_{ij} = \frac{1}{2} \cdot (x_{i',j''} + x_{i'',j'}) \quad \text{and} \quad v_i = (v_{i'} + v_{i''}).$$

It is easy to see that the weight of x equals the value of v, thereby implying that v is an optimal cover.

Lemma 1. *x is a maximum weight half-integral matching and v is an optimal cover in G.*

Proof. We will first use the fact that v' is a feasible cover for G' to show that v is a feasible cover for G. Corresponding to each edge (i, j) in G, we have two edges in G' satisfying:

$$v'_{i'} + v'_{j''} \geq \frac{1}{2} \cdot w_{ij} \quad \text{and} \quad v'_{i''} + v'_{j'} \geq \frac{1}{2} \cdot w_{ij}.$$

Therefore, in G, $v_i + v_j \geq w_{ij}$, implying feasibility of v.

By the LP-duality theorem, the weight of x' equals the value of v' in G'. Corresponding to each edge (i, j) in G, we have:

$$x_{ij} \cdot w_{ij} \ = \ \left(\frac{1}{2} \cdot (x_{i',j''} + x_{i'',j'})\right) \cdot w_{ij} \ = \ x_{i',j''} \cdot w'_{ij} + x_{i'',j'} \cdot w'_{ij}.$$

Adding over all edges, we get that the weight of x in G equals the weight of x' in G'. Furthermore, the profit of i equals the sum of profits of i' and i''. Therefore the value of v in G equals the value of v' in G'.

Putting it together, we get that the weight of w equals the value of v, implying optimality of both. Clearly, x is half-integral. The lemma follows. □

Edges that are set to half in x form connected components which are either paths or cycles. First consider a path consisting of edges $e_1, \ldots, e_k$. It contains two disjoint matchings, namely the even-numbered and odd-numbered edges. These two matching must be of equal weight, because otherwise the solution should have picked the heavier of the two matchings. Pick any of them. By the same argument, if a cycle is of even length, pick alternate edges and match them. This transforms x to a maximum weight half-integral matching in which all edges that are set to half form disjoint odd cycles. Henceforth we will assume that x satisfies this property.

Let C be a half-integral odd cycle in x of length $2k+1$, with consecutive vertices $i_1, \ldots i_{2k+1}$. Let $w_C = w_{i_1,i_2} + w_{i_2,i_3} + \ldots + w_{i_{2k+1},i_1}$ and $v_C = v_{i_1} + \ldots v_{i_{2k+1}}$. On removing any one vertex, say i_j, with its two edges from C, we are left with a path of length $2k-1$. Let M_j be the matching consisting of the k alternate edges of this path and let $w(M_j)$ be the weight of this matching.

Lemma 2. *Odd cycle C satisfies:*

1. $w_C = 2 \cdot v_C$
2. *C has a unique cover: $v_{i_j} = v_C - w(M_j)$, for $1 \leq j \leq 2k+1$.*

Proof. 1). We will use the fact that x and v are optimal solutions to LPs (5) and (6), respectively. By the primal complementary slackness condition, for $1 \leq j \leq 2k+1$, $w_{i_j,i_{j+1}} = v_{i_j} + v_{i_{j+1}}$, where addition in the subindices is done modulo $2k+1$; this follows from the fact that $x_{i_j,i_{j+1}} > 0$. Adding over all vertices of C we get $w_C = 2 \cdot v_C$.

2). By the equalities established in the proof of the first part, we get that for $1 \leq j \leq 2k+1$, $v_C = v_{i_j} + w(M_j)$. Rearranging terms gives the lemma. □

Let M' be heaviest matching among M_j, for $1 \leq j \leq 2k+1$.

Lemma 3.

$$w(M') \geq \frac{2k}{2k+1} \cdot v_C$$

Proof. Adding the equality established in the second part of Lemma 2 for all $2k+1$ values of j we get:

$$\sum_{j=1}^{2k+1} w(M_j) \ = \ (2k) \cdot v_C$$

Since M' is the heaviest of the $2k+1$ matchings in the summation, the lemma follows. □

Modify the half-integral matching x to obtain an integral matching T in G as follows. First pick all edges (i, j) such that $x_{ij} = 1$ in T. Next, for each odd cycle C, find the heaviest matching M' as described above and pick all its edges.

Definition 8. Let $1 > \alpha > 0$. A function $c : V \to \mathcal{R}_+$ is said to be an *α-approximate cover* for G if

$$\forall (i, j) \in E : \ c_i + c_j \geq \alpha \cdot w_{ij}$$

Mechanism 1. **(2/3-Approximate Core Imputation)**

1. Compute x and v, optimal solutions to LPs (5) and (6), where x is half-integral.
2. Modify x so all half-integral edges form odd cycles.
3. $\forall i \in V$, compute:

$$f_i = \begin{cases} \frac{2k}{2k+1} & \text{if } i \text{ is in a half-integral cycle of length } 2k+1. \\ 1 & \text{otherwise.} \end{cases}$$

4. $\forall i \in V$: $\quad c_i \leftarrow f_i \cdot v_i$.

Output c.

Define function $f : V \to [\frac{2}{3}, 1]$ as follows: $\forall i \in V$:

$$f_i = \begin{cases} \frac{2k}{2k+1} & \text{if } i \text{ is in a half-integral cycle of length } 2k+1. \\ 1 & \text{if } i \text{ is not in a half-integral cycle.} \end{cases}$$

Next, modify cover v to obtain an approximate cover c as follows: $\forall i \in V: \; c_i = f_i \cdot v_i$.

Lemma 4. *c is a $\frac{2}{3}$-approximate cover for G.*

Proof. Consider edge $(i, j) \in E$. Then

$$c_i + c_j \;=\; f_i \cdot v_i + f(j) \cdot v_j \geq \frac{2}{3} \cdot (v_i + v_j) \;\geq\; \frac{2}{3} \cdot w_{ij},$$

where the first inequality follows from the fact that $\forall i \in V, \;\; f_i \geq \frac{2}{3}$ and the second follows from the fact that v is a cover for G. □

The mechanism for obtaining imputation c is summarized as Mechanism 1.

Theorem 3. *The imputation c is in the $\frac{2}{3}$-approximate core of the general graph matching game.*

Proof. We need to show that c satisfies the two conditions given in Definition 5, for $\alpha = \frac{2}{3}$.

1). By Lemma 3, the weight of the matched edges picked in T from a half-integral odd cycle C of length $2k+1$ is $\geq f_k \cdot v_C \;=\; \sum_{i \in C} c(i)$. Next remove all half-integral odd cycles from G to obtain G'. Let x' and v' be the projections of x and v to G'.

By the first part of Lemma 2, the total decrease in weight in going from x to x' equals the total decrease in value in going from v to v'. Therefore, the weight of x' equals the total value of v'. Finally, observe that in G', T picks an edge (i, j) if and only if $x'_{ij} = 1$ and $\forall i \in G', \;\; c_i = v'_i$.

Adding the weight of the matching and the value of the imputation c over G' and all half-integral odd cycles we get $w(T) \geq \sum_{i \in V} c_i$.

2). Consider a coalition $S \subseteq V$. Then $p(S)$ is the weight of a maximum weight matching in G restricted to S. Assume this matching is $(i_1, j_1), \ldots (i_k, j_k)$, where $i_1, \ldots i_k$ and $j_1, \ldots j_k \in S$. Then $p(S) = (w_{i_1 j_1} + \ldots + w_{i_k j_k})$. By Lemma 4,

$$c_{i_l} + c_{j_l} \;\geq\; \frac{2}{3} \cdot w_{i_l, j_l}, \;\; \text{for } 1 \leq l \leq k.$$

Adding all k terms we get:

$$\sum_{i \in S} c_i \;\geq\; \frac{2}{3} \cdot p(S). \quad \square$$

We next show that the factor of 2/3 cannot be improved by giving a *tight example*, i.e., an infinite family of graphs on which the imputation computed is in the 2/3-approximate core.

Example 1. Consider the following infinite family of graphs. For each n, the graph G_n has $6n$ vertices i_l, j_l, k_l, for $1 \leq l \leq 2n$, and $6n$ edges $(i_l, j_l), (j_l, k_l), (i_l, k_l)$, for $1 \leq l \leq 2n$ all of weight 1. Clearly, $\mathrm{OPT}(G_n) = 2n$ and $\mathrm{OPT}_f(G_n) = 3n$. In case a connected graph is desired, add a clique on the $2n$ vertices i_l, for $1 \leq l \leq 2n$, with the weight of each edge being ϵ, where ϵ tends to zero.

Observe that for the purpose of Lemma 4, we could have defined f simply as $\forall i \in V, \ f_i = \frac{2}{3}$. However in general, this would have left a good fraction of the worth of the game unallocated. The definition of f given above improves the allocation for agents who are in large odd cycles and those who are not in odd cycles with respect to matching x. As a result, the gain of a typical sub-coalition on seceding will be less than a factor of $\frac{3}{2}$, giving it less incentive to secede. One way of formally stating an improved factor is given in Proposition 1; its proof is obvious from that of Theorem 3.

Proposition 1. *Assume that the underlying graph G has no odd cycles of length less than $2k+1$. Then imputation c is in the $\frac{2k}{2k+1}$-approximate core of the matching game for G.*

Finally, we show in Theorem 4 that the integrality gap of LP-relaxation (5) is precisely $\frac{2}{3}$. As a consequence of this fact, improving the approximation factor of an imputation for the matching game is not possible.

Theorem 4. *The integrality gap of LP-relaxation (5) is $\frac{2}{3}$.*

Proof. From the proof of the first part of Theorem 3 we get:

$$w(T) \ \geq \ \sum_{i \in V} c_i \ \geq \ \frac{2}{3} \cdot \sum_{i \in V} v_i \ = \ \frac{2}{3} \cdot w(x).$$

Therefore for any instance $I = (G, w)$,

$$\frac{\mathrm{OPT}(I)}{\mathrm{OPT}_f(I)} \geq \frac{2}{3}.$$

This places a lower bound of $\frac{2}{3}$ the integrality gap of LP-relaxation (5).

An upper bound of $\frac{2}{3}$ on the integrality gap of LP-relaxation (5) is placed by the example given in Example 1. □

5. Discussion

In the 2/3-approximate core imputation, observe that in an odd cycle of length $2k+1$, k pairs of agents are matched and one agent is left unmatched. As a consequence, monetary transfers may be needed from all $2k$ matched agents to the unmatched agent. What happens if monetary transfers to an agent are allowed from only a limited number of other agents? If so, what is the best approximation factor possible?

For the assignment game, Shapley and Shubik are able to characterize “antipodal” points in the core, i.e., imputations which are maximally distant. An analogous understanding of the $\frac{2}{3}$-approximate core of the general graph matching game will be desirable.

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